FINITE ELEMENT ANLYSIS (R18A0327)

4th Year B. Tech I- sem, Mechanical Engineering

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COURSE OBJECTIVES

UNIT - 1	CO1: To enable the students to understand fundamentals of finite element analysis and the principles involved in the discretization of domain with various elements, polynomial interpolation and assembly of global arrays.
UNIT - 2	CO2: To learn the application of FEM equations for trusses and Beams
UNIT - 3	CO3: To learn the application of FEM equations for axisymmetric problems and CST
UNIT - 4	CO4: To learn the application of FEM equations for Iso-Parametric and heat transfer problems.
UNIT - 5	CO5: To learn the application of FEM equations for dynamic analysis

UNIT 1

INTRODUCTION TO FEM & ONE-DIMENSIONAL PROBLEMS

CO1: **To enable the students to understand fundamentals of finite element analysis and the principles involved in the discretization of domain with various elements, polynomial interpolation and assembly of global arrays.**

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UNIT – I (SYLLABUS)

Introduction to FEM:

- Introduction to Finite Element Method for solving field problems, Stress and Equilibrium, Strain - Displacement relations, Stress - strain relations.

Linear Programming Problems:

- finite element modeling, local coordinates and shape functions. Potential Energy approach, Assembly of Global stiffness matrix and load vector. Finite element equations, Treatment of boundary conditions.

COURSE OUTLINE
UNIT – 1

LECTURE	LECTURE TOPIC	KEY ELEMENTS	LEARNING OBJECTIVES
1.	Introduction	Definition	Understanding of Concept of FEM (B2)
2.	Finite Element Method for solving field problems		Understanding of Concept of FEM (B2) Apply FEM Method for different fields (B3)
3.	Applications	Applications of FEM	Understanding of Applications of FEM (B3)
4.	Stress and Equilibrium Strain - Displacement relations, Stress - strain relations	Derivation	Understanding the relation between stress and strain(B2) Apply relation between stress and strain on 3D(B3)
5	finite element modeling, local coordinates and shape functions		Understanding the concept of shape functions(B2) Evaluate the shape function for 2D(B5)
6	Potential Energy approach		Understanding the concept of Rayleigh Ritz method(B2)

COURSE OUTLINE

UNIT – 1

LECTURE	LECTURE TOPIC	KEY ELEMENTS	LEARNING OBJECTIVES
6.	Assembly of Global stiffness matrix and load vector	Derivation	Understanding of Assembly of Global stiffness matrix and load vector(B2) Apply for a bar element (B3)
7.	Finite element equations, Treatment of boundary conditions.		Apply FEM Method for bar element (B3) Understanding of Treatment of boundary conditions. (B2) valuate the results for bar(B5)

LECTURE 1

Introduction to FEM

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TOPICS TO BE COVERED

- History of FEM
- Method of Engineering analysis
- Numerical Method
- Applications of FEM

LECTURE 1

Introduction FEM

INTRODUCTION TO FEM

➢The finite element analysis is a numerical technique. In this method all the complexities of the problems, like varying shape, boundary conditions and loads are maintained as they are but the solutions obtained are approximate.

➢The fast improvements in computer hardware technology and slashing of cost of computers have boosted this method, since the computer is the basic need for the application of this method.

➢A number of popular brand of finite element analysis packages are now available commercially Some of the popular packages are STAAD-PRO, GT-STRUDEL, NASTRAN, NISA and ANSYS. Using these packages one can analyze several complex structures.

METHODS OF ENGINEERING ANALYSIS

There are three methods are adopted for analyzing the product

- Experimental methods

In these methods the actual products or their proto type models or atleast their material specimen are tested by using some equipments

Ex: UTM, Rockwell hardness tester

- Analytical methods
- These methods are theoretically analyzing methods. Only simple and regular shaped products like beams, shafts, plates can be analyzed by these methods

- Numerical methods

For the products of complicated sizes and shapes with complicated material properties and boundary conditions getting solution using analytical methods is highly difficult. In such situation the numerical method can be employed

NUMERICAL METHOD

There are three numerical methods

- Functional approximating methods
- Finite element method
- Finite difference method

APPLICATIONS

S.No	Area of Study	Analysing problem
1	Civil Engineering structures	Analysis of trusses, folded plates, shell roofs, bridges and prestressed concrete structures
2	Aircraft structures	Analysis of aircraft wings, fins, rockets, space craft and missile structures
3	Mechanical Design	Stress analysis of pressure vessels, pistons, composite materials,Linkages and gears
4	Heat Conduction	Temperature distribution in solida and fluids
5	Hydraulic and water resources Engineering	Analysis of potential flows,free suface fkows,viscous flows,analysis of hydraulic structures and dams
6	Electrical Machines and Electromagnetic	Analysis of synchronous and induction machines eddy current and core losses in electric machines
7	Nuclear Engineering	Analysis of nuclear pressure vessels and containment structures
8	Geomechanics	Stress analysis in soils,dams,layered piles and machine foundations

ADVANTAGES

Using FEM we are able to

- model irregular shaped bodies quite easily
- handle general load conditions without difficulty
- model bodies composed of several different materials because the element equations are evaluated individually
- handle unlimited numbers and kinds of boundary conditions
- vary the size of the element to make it possible to use small elements
- alter the finite element model easily and cheaply
- include dynamic effects

DISADVANTAGES

- The finite element method is time consuming process
- FEM cannot produce exact results as those of analytical methods

LECTURE 2

Equations of Equilibrium for 3D Body

TOPICS TO BE COVERED

- Derivation
- Stress and strain relations
- Plane stress
- Plane strain

LECTURE 2

Equations of Equilibrium for 3D Body

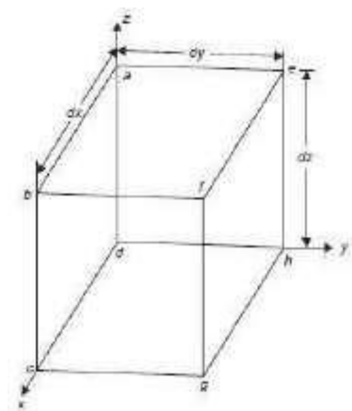
EQUATIONS OF EQUILIBRIUM FOR 3D BODY

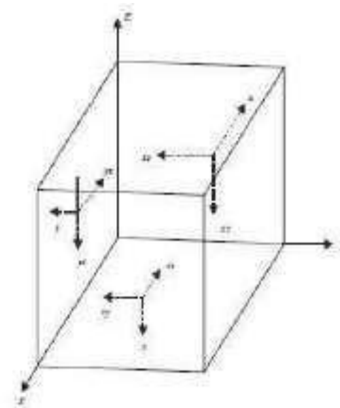
Typical three dimensional element of size dx × dy × dz. Face abcd may be called as negative face of x and the face efgh as the positive face of x since the x value for face abcd is less than that for the face efgh.

Similarly the face aehd is negative face of y and bfgc is positive face of y. Negative and positive faces of z are dhgc and aefb. The direct stresses σ and shearing stresses τ acting on the negative faces are shown in the Fig. with suitable subscript. It may be noted that the first subscript of shearing stress is the plane and the second subscript is the direction. Thus the τ_{xy} means shearing stress on the plane where x value is constant and y is the direction.

EQUATIONS OF EQUILIBRIUM FOR 3D BODY

Face	Stress on –ve Face	Stresses on +ve Face
x	σ_x	$\sigma_x^+ = \sigma_x + \frac{\partial \sigma_x}{\partial x} dx$
	τ_{xy}	$\tau_{xy}^+ = \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx$
	τ_{xz}	$\tau_{xz}^+ = \tau_{xz} + \frac{\partial \tau_{xz}}{\partial x} dx$
y	σ_y	$\sigma_y^+ = \sigma_y + \frac{\partial \sigma_y}{\partial y} dy$
	τ_{yx}	$\tau_{yx}^+ = \tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy$
	τ_{yz}	$\tau_{yz}^+ = \tau_{yz} + \frac{\partial \tau_{yz}}{\partial y} dy$
z	σ_z	$\sigma_z^+ = \sigma_z + \frac{\partial \sigma_z}{\partial z} dz$
	τ_{zx}	$\tau_{zx}^+ = \tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz$
	τ_{zy}	$\tau_{zy}^+ = \tau_{zy} + \frac{\partial \tau_{zy}}{\partial z} dz$





EQUATIONS OF EQUILIBRIUM FOR 3D BODY

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial_z} + X = 0$$

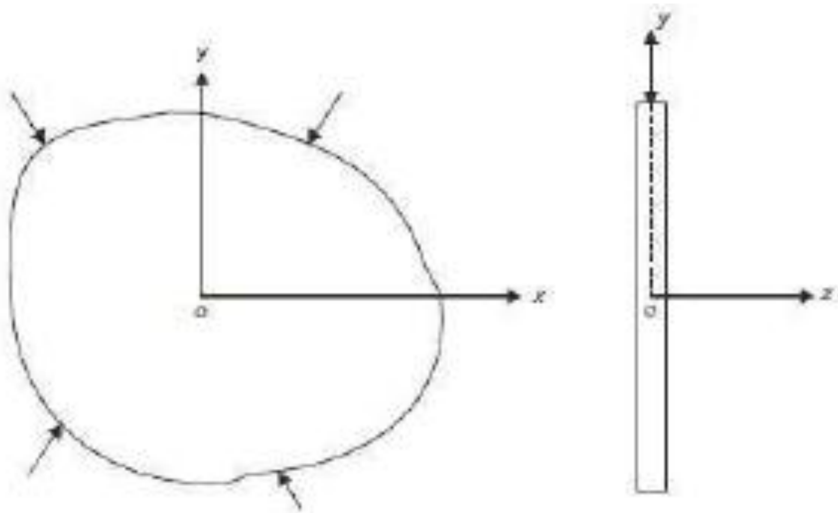
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0$$

$\tau_{xy} = \tau_{yx}$, $\tau_{yz} = \tau_{zy}$ and $\tau_{xz} = \tau_{zx}$

PLANE STRESS PROBLEM

The thin plates subject to forces in their plane only, fall under this category of the problems. Fig. shows a typical plane stress problem. In this, there is



no force in the z-direction and no variation of any forces in z-direction. Hence

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

The conditions $\tau_{xz} = \tau_{yz} = 0$ give $\gamma_{xz} = \gamma_{yz} = 0$ and the condition $\sigma_z = 0$ gives,

$$\sigma_z = \mu e_x + \mu e_y + (1-\mu) e_z = 0$$

PLANE STRESS PROBLEM

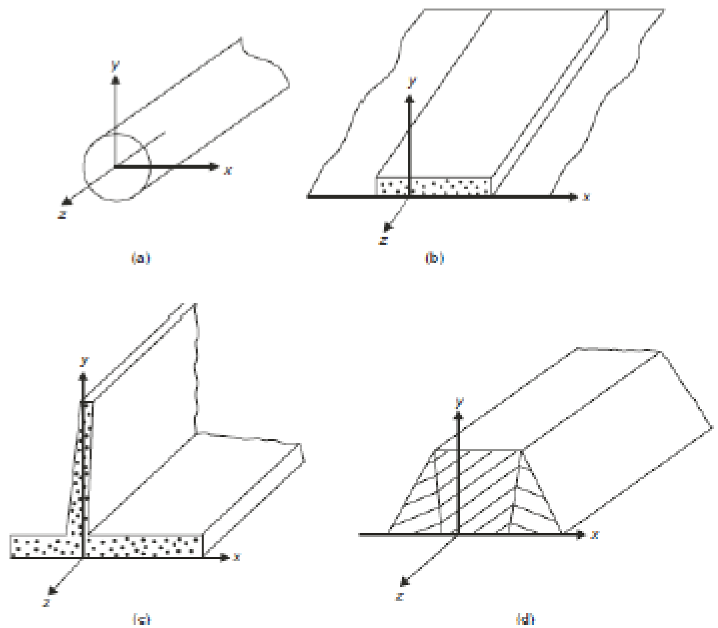
i.e. $$\varepsilon_z = -\frac{\mu}{1-\mu}\left(\varepsilon_x + \varepsilon_y\right)$$

If this is substituted in equation 2.13 the constitutive law reduces to

$$\begin{Bmatrix}\sigma_x \\ \sigma_y \\ \tau_{xy}\end{Bmatrix} = \frac{E}{1-\mu^2}\begin{bmatrix}1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}\end{Bmatrix}$$

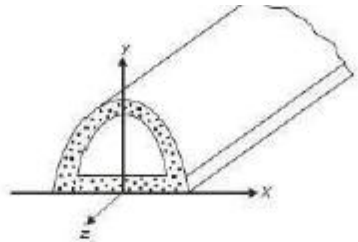
PLANE STRAIN PROBLEM

A long body subject to significant lateral forces but very little longitudinal forces falls under this category of problems. Examples of such problems are pipes, long strip footings, retaining walls, gravity dams, tunnels, etc. In these problems, except for a small distance at the ends, state of stress is represented by any small longitudinal strip. The displacement in longitudinal direction (z-direction) is zero in typical strip



DEPARTMENT OF MECHANICAL ENGINEERING

PLANE STRAIN PROBLEM



(a)

Fig. 2.7 *(contd)*

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

$\gamma_{xz} = \gamma_{yz} = 0$ means τ_{xz} and τ_{yz} are zero.

$\varepsilon_z = 0$ means

$$\varepsilon_z = \frac{\sigma_z}{E} - \mu \frac{(\sigma_x + \sigma_y)}{E} = 0$$

i.e. $$\sigma_z = \mu(\sigma_x + \sigma_y)$$

Hence equation 2.13 when applied to plane strains problems reduces to

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{pmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \frac{1-2\mu}{2} \end{pmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

TOPICS TO BE COVERED

- Rayleigh Ritz method

LECTURE 3

A generic problem in 1D

$$\frac{d^2u}{dx^2} + x = 0; \qquad 0 < x < 1$$

$$u = 0 \quad at \; x = 0$$

$$u = 1 \quad at \; x = 1$$

Approximate solution strategy:

Guess $u(x) = a_0\varphi_o(x) + a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots$

Where $\varphi_o(x)$, $\varphi_1(x)$,… are "known" functions and a_o, a_1, etc are constants chosen such that the approximate solution

1. Satisfies the boundary conditions
2. Satisfies the differential equation

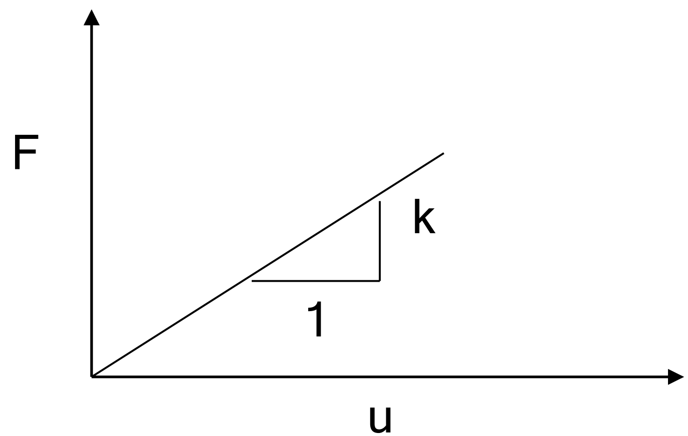
Too difficult to satisfy for general problems!!

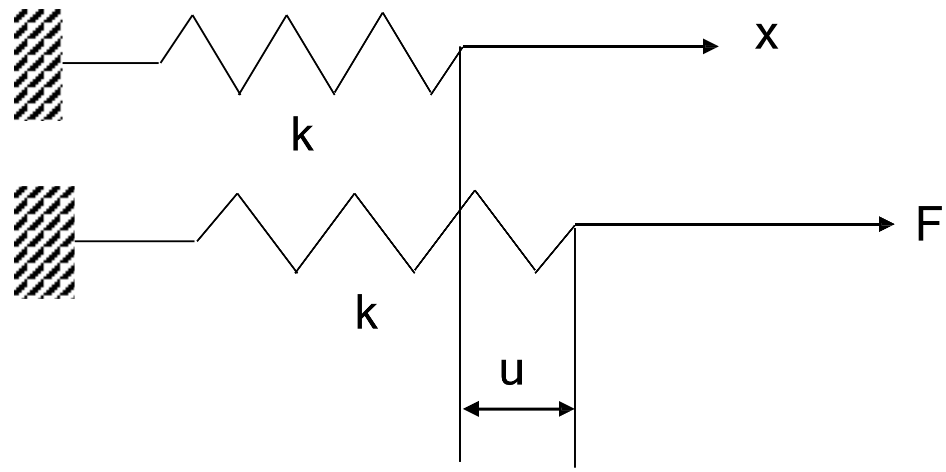
Potential energy

The **potential energy** of an elastic body is defined as

$$\Pi = \text{Strain energy (U)} - \text{potential energy of loading } (W)$$

Strain energy of a linear spring

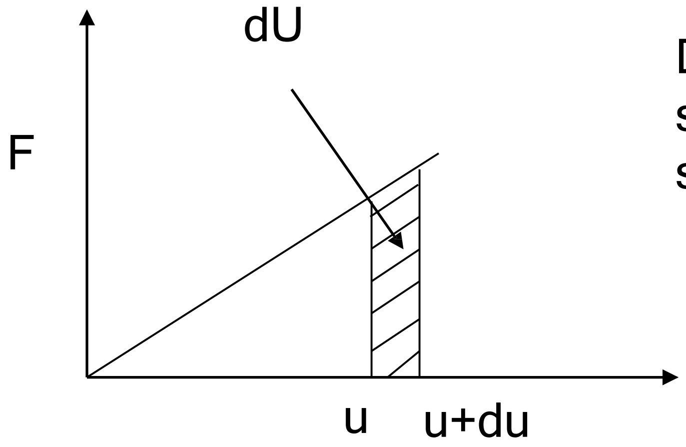




F = Force in the spring
u = deflection of the spring
k = “stiffness” of the spring

Hooke’s Law
F = ku

Strain energy of a linear spring



Differential strain energy of the spring for a small change in displacement (du) of the spring

$$dU = F du$$

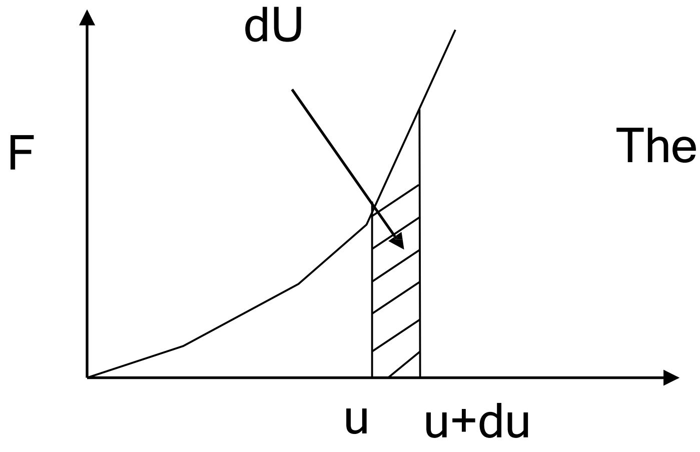
For a <u>linear spring</u>

$$dU = k u du$$

The total strain energy of the spring

$$U = \int_0^u k\, u\, du = \frac{1}{2} k\, u^2$$

Strain energy of a nonlinear spring



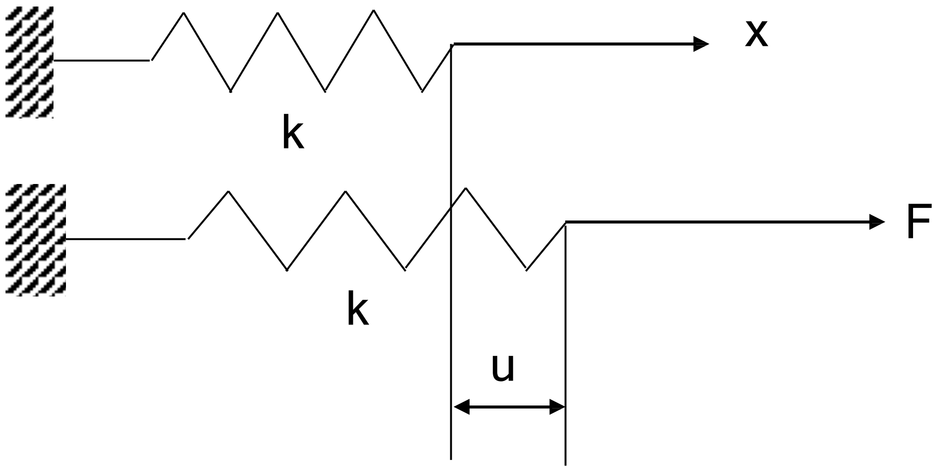
$$dU = F du$$

The total strain energy of the spring

$$U = \int_0^u F\, du = \text{Area under the force} - \text{dispalcement curve}$$

Potential energy of the loading (for a single spring as in the figure)

$$W = Fu$$

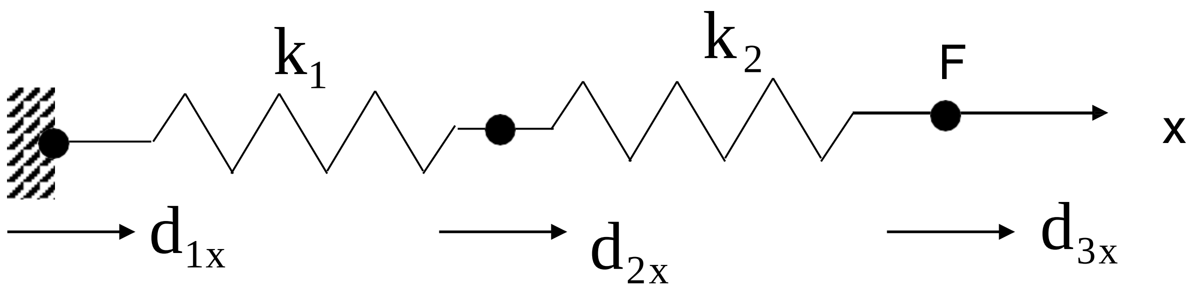


Potential energy of a linear spring

$$\Pi = \text{Strain energy (U)} - \text{potential energy of loading (W)}$$

$$\Pi = \frac{1}{2}ku^2 - Fu$$

Principle of <u>minimum potential energy</u> for a system of springs



For this system of spring, first write down the total potential energy of the system as:

$$\Pi_{system} = \left[\frac{1}{2}k_1(d_{2x})^2 + \frac{1}{2}k_2(d_{3x} - d_{2x})^2\right] - Fd_{3x}$$

Obtain the equilibrium equations by minimizing the potential energy

$$\frac{\partial \Pi_{system}}{\partial d_{2x}} = k_1 d_{2x} - k_2(d_{3x} - d_{2x}) = 0 \qquad Equation\,(1)$$

$$\frac{\partial \Pi_{system}}{\partial d_{3x}} = k_2(d_{3x} - d_{2x}) - F = 0 \qquad Equation\,(2)$$

Principle of minimum potential energy for a system of springs

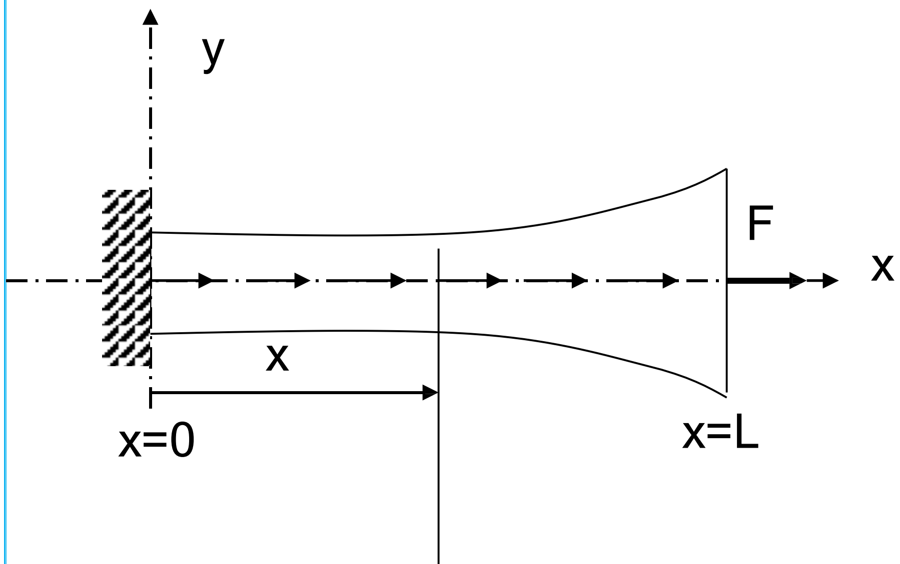
In matrix form, equations 1 and 2 look like

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} d_{2x} \\ d_{3x} \end{bmatrix} = \begin{bmatrix} 0 \\ F \end{bmatrix}$$

Does this equation look familiar?

Also look at example problem **worked out in class**

Axially loaded elastic bar



A(x) = cross section at x
b(x) = body force distribution (force per unit length)
E(x) = Young's modulus
u(x) = displacement of the bar at x

Axial strain $\varepsilon = \dfrac{du}{dx}$

Axial stress $\sigma = E\varepsilon = E\dfrac{du}{dx}$

Strain energy per unit volume of the bar $dU = \dfrac{1}{2}\sigma\varepsilon = \dfrac{1}{2}E\left(\dfrac{du}{dx}\right)^2$

Strain energy of the bar

$$U = \int dU = \int \frac{1}{2}\sigma\varepsilon\, dV = \int^{L} \frac{1}{2}\sigma\varepsilon\, A dx$$

since dV=Adx

Axially loaded elastic bar

Strain energy of the bar

$$U = \int_0^L \frac{1}{2} \sigma \varepsilon A \, dx = \frac{1}{2} \int_0^L EA \left(\frac{du}{dx} \right)^2 dx$$

Potential energy of the loading

$$W = \int_0^L bu \, dx + Fu(x = L)$$

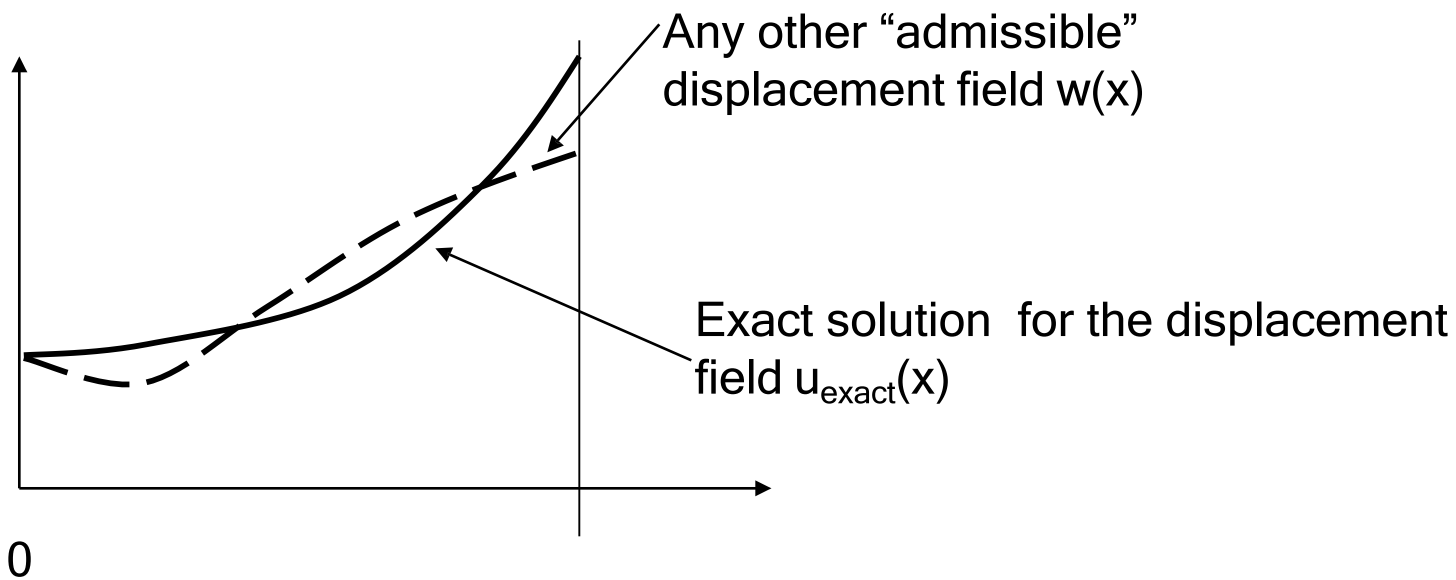
Potential energy of the axially loaded bar

$$\Pi = \frac{1}{2} \int_0^L EA \left(\frac{du}{dx} \right)^2 dx - \int_0^L bu \, dx - Fu(x = L)$$

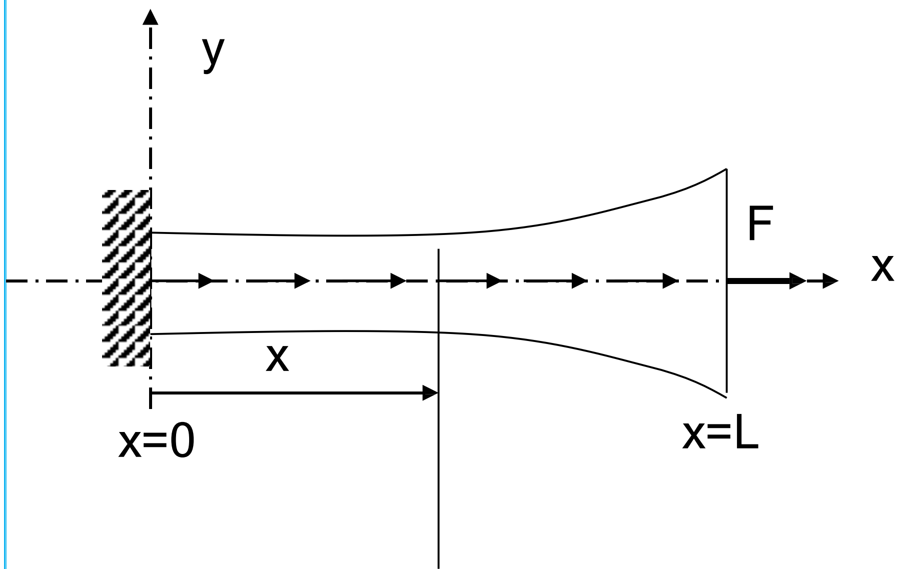
Principle of Minimum Potential Energy

Among all admissible displacements that a body can have, the one that minimizes the total potential energy of the body satisfies the strong formulation

Admissible displacements: these are any reasonable displacement that you can think of that satisfy the ***displacement boundary conditions of the original problem*** (and of course certain minimum ***continuity requirements***). Example:



Lets see what this means for an axially loaded elastic bar



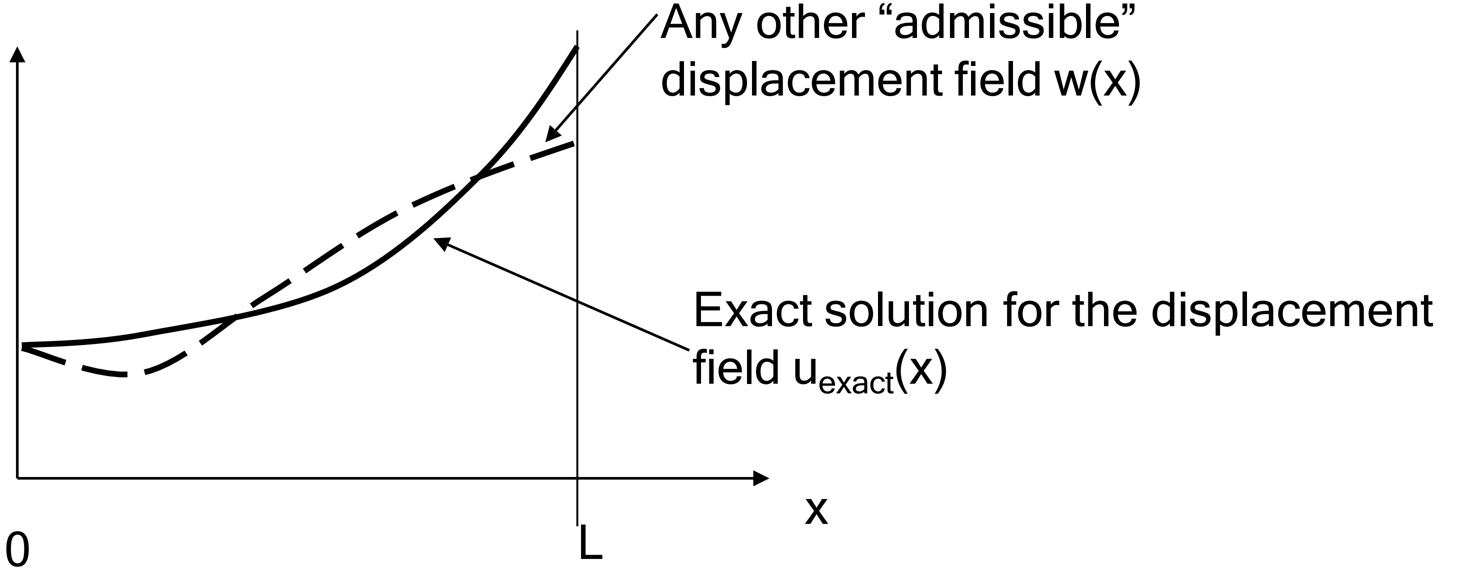
A(x) = cross section at x
b(x) = body force distribution (force per unit length)
E(x) = Young's modulus

Potential energy of the axially loaded bar corresponding to the exact solution $u_{exact}(x)$

$$\Pi(u_{exact}) = \frac{1}{2}\int_0^L EA\left(\frac{du_{exact}}{dx}\right)^2 dx - \int_0^L b u_{exact} \, dx - F u_{exact}(x = L)$$

Potential energy of the axially loaded bar corresponding to the "**admissible**" displacement w(x)

$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\ dx - Fw(x = L)$$



Example:

$$AE\frac{d^2u}{dx^2}+b=0; \qquad 0<x<L$$

$$u=0 \quad at \quad x=0$$

$$EA\frac{du}{dx}=F \quad at \ x=L$$

Assume EA=1; b=1; L=1; F=1

Analytical solution is

$$u_{exact}=2x-\frac{x^2}{2}$$

Potential energy corresponding to this analytical solution

$$\Pi(u_{exact})=\frac{1}{2}\int_0^1\left(\frac{du_{exact}}{dx}\right)^2 dx-\int_0^1 u_{exact}\ dx-u_{exact}(x=1)=-\frac{7}{6}$$

Now assume an admissible displacement

$$w = x$$

Why is this an "**admissible**" displacement? This displacement is quite arbitrary. But, it satisfies the given **displacement boundary condition** w(x=0)=0. Also, its first derivate does not blow up.

Potential energy corresponding to this admissible displacement

$$\Pi(w) = \frac{1}{2}\int_0^1 \left(\frac{dw}{dx}\right)^2 dx - \int_0^1 w \, dx - w(x=1) = -1$$

Notice

since $-\frac{7}{6} < -1$

$$\boxed{\Pi(u_{exact}) < \Pi(w)}$$

Principle of Minimum Potential Energy

Among all admissible displacements that a body can have, the one that minimizes the total potential energy of the body satisfies the strong formulation

Mathematical statement: If 'u_{exact}' is the exact solution (which satisfies the differential equation together with the boundary conditions), and 'w' is an admissible displacement (that is quite arbitrary except for the fact that it **satisfies the displacement boundary conditions** and its **first derivative does not blow up**), then

$$\Pi(u_{exact}) < \Pi(w)$$

unless $w = u_{exact}$ (i.e. **the exact solution minimizes the potential energy**)

The Principle of Minimum Potential Energy and the strong formulation are exactly equivalent statements of the same problem.

The exact solution (u_{exact}) that satisfies the strong form, renders the potential energy of the system a minimum.

So, why use the Principle of Minimum Potential Energy?

The short answer is that it is much less demanding than the strong formulation.

The long answer is, it

1. requires only the first derivative to be finite
2. incorporates the force boundary condition automatically. The admissible displacement (which is the function that you need to choose) needs to satisfy only the displacement boundary condition

Finite element formulation, takes as its starting point, not the strong formulation, but the **Principle of Minimum Potential Energy**.

Task is to find the function 'w' that minimizes the potential energy of the system

$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\, dx - Fw(x = L)$$

From the Principle of Minimum Potential Energy, that function 'w' is the exact solution.

Rayleigh-Ritz Principle

The minimization of the potential energy is difficult to perform exactly.
The Rayleigh-Ritz principle is an **approximate** way of doing this.

Step 1. Assume a solution

$$w(x) = a_0 \varphi_o(x) + a_1 \varphi_1(x) + a_2 \varphi_2(x) + \ldots$$

Where $\varphi_o(x)$, $\varphi_1(x)$,… are "***admissible***" functions and a_o, a_1, etc are constants to be determined from the solution.

Rayleigh-Ritz Principle

Step 2. Plug the approximate solution into the potential energy

$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\, dx - Fw(x = L)$$

$$\Rightarrow \Pi(a_0, a_1, \ldots) = \frac{1}{2}\int_0^L EA\left(a_0\frac{d\varphi_0}{dx} + a_1\frac{d\varphi_1}{dx} + \ldots\right)^2 dx$$

$$- \int_0^L b\left(a_0\varphi_0 + a_1\varphi_1 + \ldots\right)dx$$

$$- F\left(a_0\varphi_0(x = L) + a_1\varphi_1(x = L) + \ldots\right)$$

Rayleigh-Ritz Principle

Step 3. Obtain the coefficients a_o, a_1, etc by setting

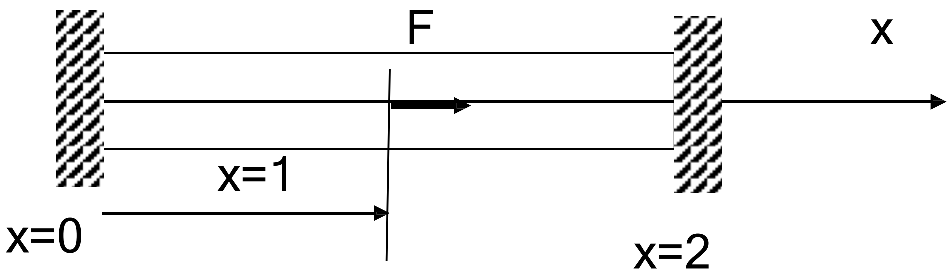
$$\frac{\partial \Pi(\mathrm{w})}{\partial a_i} = 0, \quad i = 0,1,2,...$$

The approximate solution is

$$u(x) = a_0 \varphi_o(x) + a_1 \varphi_1(x) + a_2 \varphi_2(x) + ...$$

Where the coefficients have been obtained from step 3

Example of application of Rayleigh Ritz Principle



E=A=1
F=2

The potential energy of this bar (of length 2) is

$$\Pi(u) = \underbrace{\frac{1}{2}\int_0^2 \left(\frac{du}{dx}\right)^2 dx}_{StrainEnergy} - \underbrace{Fu(x=1)}_{\substack{PotentialEnergy \\ of\ load\ F\ applied \\ at\ x=1}}$$

Let us assume a polynomial “admissible” displacement field

$$u = a_0 + a_1 x + a_2 x^2$$

Note that this is NOT the analytical solution for this problem.

Example of application of Rayleigh Ritz Principle

For this "admissible" displacement to satisfy the **displacement boundary conditions** the following conditions must be satisfied:

$$\mathrm{u}(\mathrm{x}=0)=a_0=0$$
$$\mathrm{u}(\mathrm{x}=2)=a_0+2a_1+4a_2=0$$

Hence, we obtain

$$a_0=0$$
$$a_1=-2a_2$$

Hence, the "admissible" displacement simplifies to

$$\begin{aligned}\mathrm{u}&=a_0+a_1x+a_2x^2\\&=a_2\left(-2x+x^2\right)\end{aligned}$$

Now we apply **Rayleigh Ritz principle**, which says that if I plug this approximation into the expression for the potential energy Π, I can obtain the unknown (in this case a_2) by minimizing Π

$$\Pi(u) = \frac{1}{2}\int_0^2 \left(\frac{du}{dx}\right)^2 dx - Fu(x=1)$$

$$= \frac{1}{2}\int_0^2 \left(\frac{d}{dx}\left\{a_2\left(-2x+x^2\right)\right\}\right)^2 dx - F\left\{a_2\left(-2x+x^2\right)\right\}_{\substack{evaluated \\ at\ x=1}}$$

$$= \frac{4}{3}a_2^{\ 2} + 2a_2$$

$$\frac{\partial \Pi}{\partial a_2} = 0$$

$$\Rightarrow \frac{8}{3}a_2 + 2 = 0$$

$$\Rightarrow a_2 = -\frac{3}{4}$$

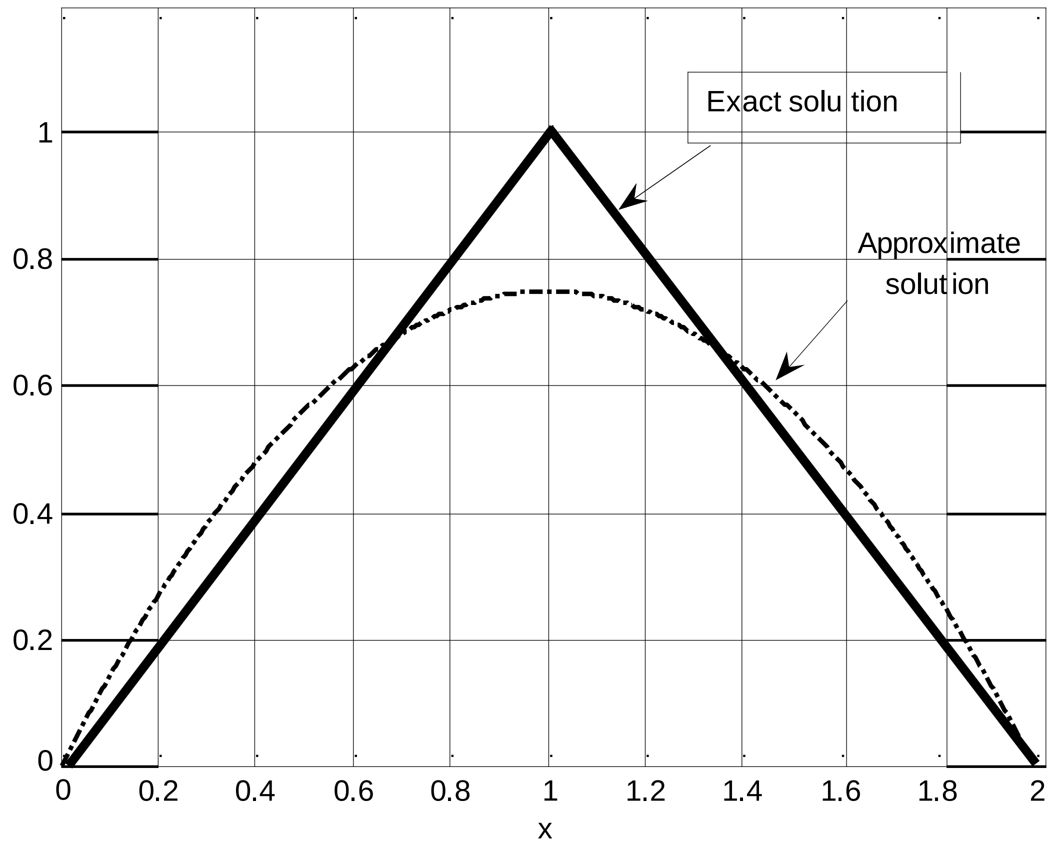
Hence the approximate solution to this problem, using the Rayleigh-Ritz principle is

$$\begin{aligned} \mathrm{u} &= a_0 + a_1 x + a_2 x^2 \\ &= a_2\left(-2x + x^2\right) \\ &= -\frac{3}{4}\left(-2x + x^2\right) \end{aligned}$$

Notice that the exact answer to this problem (can you prove this?) is

$$\mathrm{u}_{\mathrm{exact}} = \begin{cases} x & for \;\; 0 \le x < 1 \\ 2 - x & for \;\; 1 \le x \le 2 \end{cases}$$

The **displacement** solution :



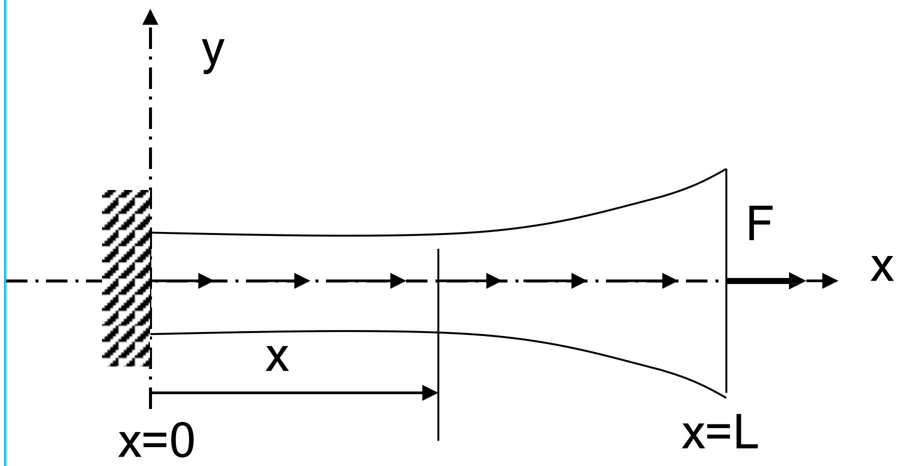
How can you improve the approximation?

TOPICS TO BE COVERED

- Shape functions
- One dimensional problems

LECTURE 4

Axially loaded elastic bar



A(x) = cross section at x

b(x) = body force distribution (force per unit length)

E(x) = Young's modulus

Potential energy of the axially loaded bar corresponding to the exact solution u(x)

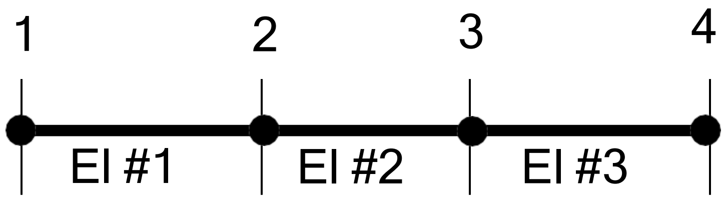
$$\Pi(u) = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - \int_0^L bu\, dx - Fu(x = L)$$

Potential energy of the bar corresponding to an **admissible displacement** w(x)

$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\, dx - Fw(x = L)$$

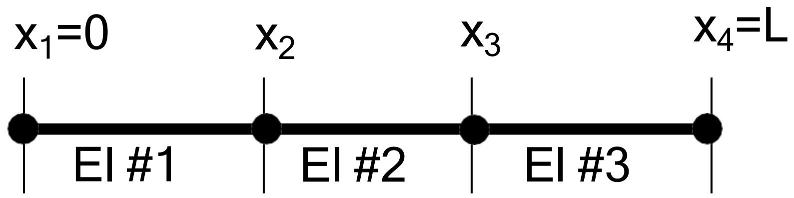
Finite element idea:

Step 1: Divide the truss into **finite elements** connected to each other through special points ("**nodes**")



Total potential energy=sum of potential energies of the elements

$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\ dx - Fw(x = L)$$



Total potential energy

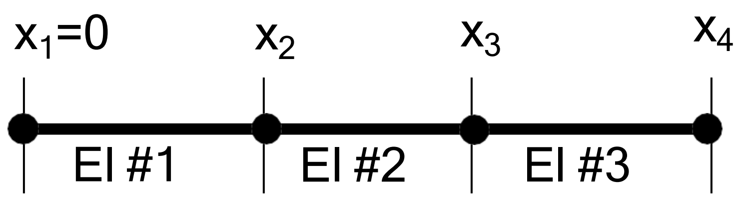
$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\,dx - Fw(x=L)$$

Potential energy of element 1:

$$\Pi_1(w) = \frac{1}{2}\int_{x_1}^{x_2} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_1}^{x_2} bw\,dx$$

Potential energy of element 2:

$$\Pi_2(w) = \frac{1}{2}\int_{x_2}^{x_3} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_2}^{x_3} bw\,dx$$



Potential energy of element 3:

$$\Pi_3(w) = \frac{1}{2}\int_{x_3}^{x_4} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_3}^{x_4} bw\, dx - Fw(x = L)$$

Total potential energy=sum of potential energies of the elements

$$\Pi(w) = \Pi_1(w) + \Pi_2(w) + \Pi_3(w)$$

Step 2: Describe the behavior of each element

In the "**direct stiffness**" approach, we derived the stiffness matrix of each element directly (See lecture on Springs/Trusses).

Now, we will first approximate the **displacement** inside each element and then show you a systematic way of deriving the stiffness matrix (sections 2.2 and 3.1 of Logan).

TASK 1: APPROXIMATE THE DISPLACEMENT WITHIN EACH ELEMENT
TASK 2: APPROXIMATE THE STRAIN and STRESS WITHIN EACH ELEMENT
TASK 3: DERIVE THE STIFFNESS MATRIX OF EACH ELEMENT (this class) USING THE RAYLEIGH-RITZ PRINCIPLE

Summary

Inside an element, the three most important approximations **in terms of the nodal displacements** ($\underline{d}$) are:

Displacement approximation in terms of shape functions

$$w(x) = \underline{N}\,\underline{d} \tag{1}$$

Strain approximation in terms of strain-displacement matrix

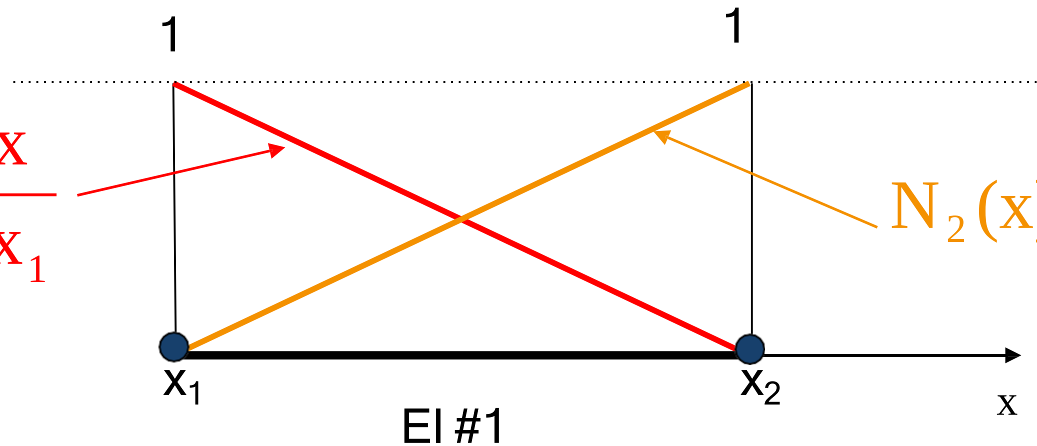
$$\varepsilon(x) = \underline{B}\,\underline{d} \tag{2}$$

Stress approximation in terms of strain-displacement matrix and Young's modulus

$$\sigma = E\underline{B}\,\underline{d} \tag{3}$$

The shape functions for a **1D linear element**

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$



$$N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

Within the element, the displacement approximation is

$$w(x) = \frac{x_2 - x}{x_2 - x_1} d_{1x} + \frac{x - x_1}{x_2 - x_1} d_{2x}$$

For a **linear element**

Displacement approximation in terms of shape functions

$$w(x) = \begin{bmatrix} \frac{x_2 - x}{x_2 - x_1} & \frac{x - x_1}{x_2 - x_1} \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

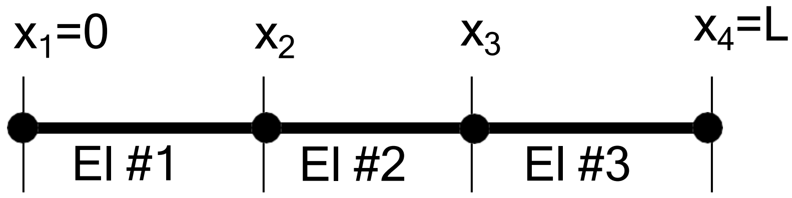
Strain approximation

$$\varepsilon = \frac{dw}{dx} = \frac{1}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

Stress approximation

$$\sigma = E\varepsilon = \frac{E}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

Why is the approximation "admissible"?



For the entire bar, the displacement approximation is

$$w(x) = w^{(1)}(x) + w^{(2)}(x) + w^{(3)}(x)$$

Where $w^{(i)}(x)$ is the displacement approximation within element (i).
Let use set $d_{1x}=0$. Then, can you seen that the above approximation does satisfy the two conditions of being an admissible function on the entire bar, i.e.,

$$(1)\ w(x=0)=0$$

$$(2)\ \frac{dw}{dx}\ \text{exists}$$

TASK 3: DERIVE THE STIFFNESS MATRIX OF EACH ELEMENT USING THE RAYLEIGH-RITZ PRINCIPLE

Potential energy of element 1:

$$\Pi_1(w) = \frac{1}{2}\int_{x_1}^{x_2} \sigma \quad \varepsilon \quad A dx - \int_{x_1}^{x_2} bw \quad dx$$

Lets plug in the **approximation**

$$w(x) = \underline{N}\,\underline{d}$$

$$\varepsilon(x) = \underline{B}\,\underline{d}$$

$$\sigma = E\underline{B}\,\underline{d}$$

$$\Pi_1(\underline{d}) = \frac{1}{2}\underline{d}^T \left(\int_{x_1}^{x_2} \underline{B}^T E \underline{B} \quad A dx\right)\underline{d} - \underline{d}^T \left(\int_{x_1}^{x_2} \underline{N}^T \quad b \quad dx\right)$$

Lets see what the matrix

$$\int_{x_1}^{x_2} \underline{B}^T E \underline{B} \; A dx$$

is for a 1D linear element

Recall that

$$\boxed{\underline{B} = \frac{1}{x_2 - x_1}\begin{bmatrix}-1 & 1\end{bmatrix}}$$

Hence

$$\underline{B}^T \; E \; \underline{B} = \frac{1}{x_2 - x_1}\begin{Bmatrix}-1 \\ 1\end{Bmatrix} E \frac{1}{x_2 - x_1}\begin{bmatrix}-1 & 1\end{bmatrix}$$

$$= \frac{E}{(x_2 - x_1)^2}\begin{Bmatrix}-1 \\ 1\end{Bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix} = \frac{E}{(x_2 - x_1)^2}\begin{bmatrix}1 & -1 \\ -1 & 1\end{bmatrix}$$

$$\int_{x_1}^{x_2} \underline{B}^T E \underline{B} \; A dx = \frac{1}{(x_2 - x_1)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_{x_1}^{x_2} AE dx = \left(\int_{x_1}^{x_2} AE dx \right) \frac{1}{(x_2 - x_1)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Now, if we **assume** E and A are constant

$$\int_{x_1}^{x_2} \underline{B}^T E \underline{B} \; A dx = \left(\int_{x_1}^{x_2} AE dx \right) \frac{1}{(x_2 - x_1)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{AE(x_2 - x_1)}{(x_2 - x_1)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{AE}{(x_2 - x_1)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Remembering that $(x_2$-$x_1)$ is the length of the element, this is the **stiffness matrix** we had derived directly before using the **direct stiffness** approach!!

Then why is it necessary to go through this complicated procedure??

1. Easy to handle **nonuniform** E and A
2. Easy to handle **distributed loads**

For nonuniform E and A, i.e. E(x) and A(x), the **stiffness matrix** of the linear element will **NOT** be

$$\frac{EA}{(x_2 - x_1)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

But it will **ALWAYS be**

$$\underline{k} = \int_{x_1}^{x_2} \underline{B}^T E \underline{B} \; A dx$$

Now lets go back to

$$\Pi_1(\underline{\mathrm{d}}) = \frac{1}{2}\underline{\mathrm{d}}^T \left(\underbrace{\int_{x_1}^{x_2} \underline{\mathrm{B}}^{\mathrm{T}} \mathrm{E} \underline{\mathrm{B}} \; A \mathrm{dx}}_{\underline{k}} \right) \underline{\mathrm{d}} - \underline{\mathrm{d}}^T \left(\underbrace{\int_{x_1}^{x_2} \underline{\mathrm{N}}^T \; \mathrm{b} \; \mathrm{dx}}_{\underline{f_b}} \right)$$

$$= \frac{1}{2}\underline{\mathrm{d}}^T \underline{k}\,\underline{\mathrm{d}} - \underline{\mathrm{d}}^T \underline{f_b}$$

Element **stiffness matrix**

$$\boxed{\underline{k} = \int_{x_1}^{x_2} \underline{\mathrm{B}}^{\mathrm{T}} \mathrm{E} \underline{\mathrm{B}} \; A \mathrm{dx}}$$

Element **nodal load vector due to distributed body force**

$$\boxed{\underline{f_b} = \int_{x_1}^{x_2} \underline{\mathrm{N}}^T \; \mathrm{b} \, \mathrm{dx}}$$

Apply Rayleigh-Ritz principle for the 1D linear element

$$\left.\begin{aligned}\frac{\partial \Pi_1(\underline{d})}{\partial d_{1x}} = 0 \\ \frac{\partial \Pi_1(\underline{d})}{\partial d_{2x}} = 0\end{aligned}\right\} \Rightarrow \frac{\partial \Pi_1(\underline{d})}{\partial \underline{d}} = 0$$

Recall from linear algebra (Lecture notes on Linear Algebra)

$$\Pi_1(\underline{d}) = \frac{1}{2}\underline{d}^T \underline{k}\,\underline{d} - \underline{d}^T \underline{f_b}$$

$$\Rightarrow \frac{\partial \Pi_1(\underline{d})}{\partial \underline{d}} = \underline{k}\underline{d} - \underline{f_b}$$

Hence

$$\frac{\partial \Pi_1(\underline{\mathrm{d}})}{\partial \underline{\mathrm{d}}} = 0$$

$$\boxed{\Rightarrow \underline{k}\mathrm{d} = \underline{f_b}}$$

Exactly the same equation that we had before, except that the stiffness matrix and nodal force vectors are more general

Recap of the properties of the element stiffness matrix

$$\underline{k} = \int_{x_1}^{x_2} \underline{B}^T E \underline{B} \; A dx$$

1. The stiffness matrix is **singular** and is therefore non-invertible
2. The stiffness matrix is **symmetric**
3. Sum of any row (or **column**) of the stiffness matrix is zero!

k_{11}

Why?

Sum of any row (or column) of the stiffness matrix is zero

Consider a **rigid body motion** of the element

$$\underline{d} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Element strain $\varepsilon = 0 = \underline{B}\ \underline{d}$

$$\Rightarrow \underline{k}\ \underline{d} = \left(\int_{x_1}^{x_2} \underline{B}^T E \underline{B}\ A dx \right) \underline{d}$$

$$= \int_{x_1}^{x_2} \underline{B}^T E \left(\underline{B}\underline{d} \right) A dx$$

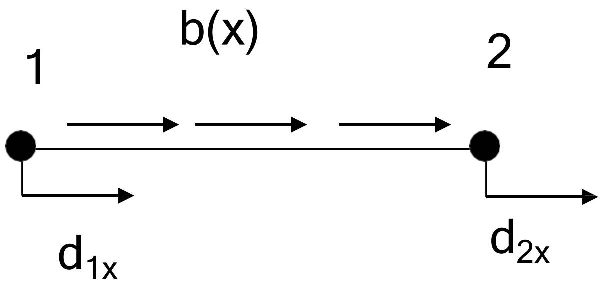
$$= \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\underline{k}\ \underline{d} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow k_{11} + k_{12} = 0 \quad \text{and} \quad k_{21} + k_{22} = 0$$

The nodal load vector

$$\underline{f}_b = \int_{x_1}^{x_2} \underline{N}^T \ b \ dx$$



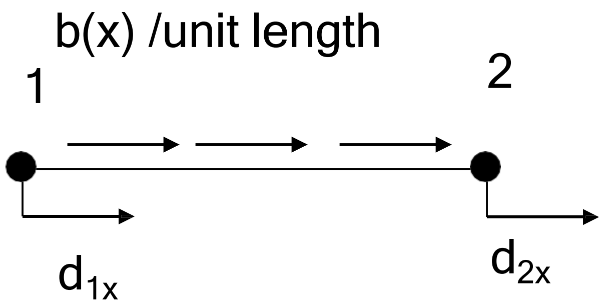
$$\underline{f}_b = \int_{x_1}^{x_2} \underline{N} \ b \ dx = \int_{x_1}^{x_2} \begin{Bmatrix} N_1(x) \\ N_2(x) \end{Bmatrix} b \ dx$$

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{Bmatrix} \int_{x_1}^{x_2} N_1(x) \ b \ dx \\ \int_{x_1}^{x_2} N_2(x) \ b \ dx \end{Bmatrix}$$

$$f_{1x} = \int_{x_1}^{x_2} N_1(x) \ b \ dx$$

$$f_{2x} = \int_{x_1}^{x_2} N_2(x) \ b \ dx$$

"**Consistent**" nodal loads



Replaced by

A distributed load is represented by two nodal loads in a consistent manner

e.g., if b=1

$$f_{1x} = \int_{x_1}^{x_2} N_1(x)\ \mathrm{b\ dx} = \int_{x_1}^{x_2} N_1(x)\ \ \mathrm{dx} = \frac{\mathrm{x}_2 - x_1}{2}$$

$$f_{2x} = \int_{x_1}^{x_2} N_2(x)\ \mathrm{b\ dx} = \int_{x_1}^{x_2} N_2(x)\ \ \mathrm{dx} = \frac{\mathrm{x}_2 - x_1}{2}$$

Divide the **total force** into two equal halves and lump them at the nodes

What happens if b(x)=x?

Summary: For each element

Displacement approximation in terms of shape functions

$$w(x) = \underline{N}\ \underline{d}$$

Strain approximation in terms of strain-displacement matrix

$$\varepsilon(x) = \underline{B}\ \underline{d}$$

Stress approximation

$$\sigma = E\underline{B}\ \underline{d}$$

Element stiffness matrix

$$\underline{k} = \int_{x_1}^{x_2} \underline{B}^T E \underline{B}\ A dx$$

Element nodal load vector

$$\underline{f_b} = \int_{x_1}^{x_2} \underline{N}^T\ b\, dx$$

What happens for element #3?

$$\Pi_3(w) = \frac{1}{2}\int_{x_3}^{x_4} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_3}^{x_4} bw\, dx - Fw(x = L)$$

For element 3

$$w(x) = \begin{bmatrix} \frac{x_4 - x}{x_4 - x_3} & \frac{x - x_3}{x_4 - x_3} \end{bmatrix} \begin{Bmatrix} d_{3x} \\ d_{4x} \end{Bmatrix}$$

$$\Rightarrow w(x = L) = d_{4x}$$

The discretized form of the potential energy

$$\Pi_3(\underline{d}) = \frac{1}{2}\underline{d}^T\left(\int_{x_3}^{x_4} \underline{B}^T E \underline{B}\ A dx\right)\underline{d} - \underline{d}^T\left(\int_{x_3}^{x_4} \underline{N}^T\ b\ dx\right) - Fd_{4x}$$

What happens for element #3?

Now apply Rayleigh-Ritz principle

$$\frac{\partial \Pi_3(\underline{d})}{\partial \underline{d}} = 0$$

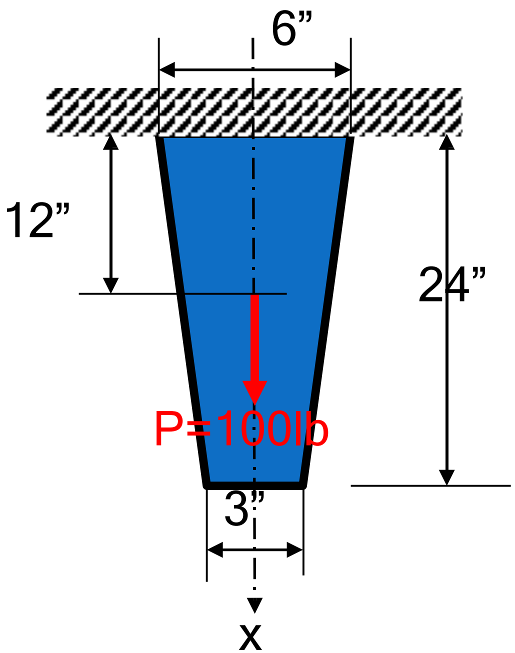
$$\Rightarrow \underline{kd} = \underline{f_b} + \begin{Bmatrix} 0 \\ F \end{Bmatrix}$$

Hence there is an extra load term on the right hand side due to the concentrated force F applied to the right end of the bar.

NOTE that whenever you have a concentrated load at ANY node, that load should be applied as an extra right hand side term.

Step3:Assembly exactly as you had done before, assemble the global stiffness matrix and global load vector and solve the resulting set of equations by properly taking into account the displacement boundary conditions

Problem:



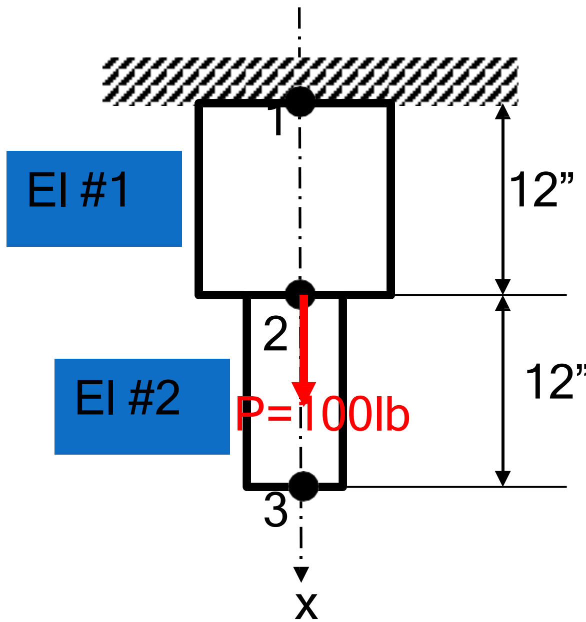
$E=30x10^6$ psi
$\rho=0.2836$ lb/in^3
Thickness of plate, t=1"

Model the plate as 2 finite elements and

(1) Write the expression for element stiffness matrix and body force vectors
(2) Assemble the global stiffness matrix and load vector
(3) Solve for the unknown displacements
(4) Evaluate the stress in each element
(5) Evaluate the reaction in each support

Solution (1)

Finite element model



Node-element connectivity chart

Element #	Node 1	Node 2
1	1	2
2	2	3

Stiffness matrix of El #1

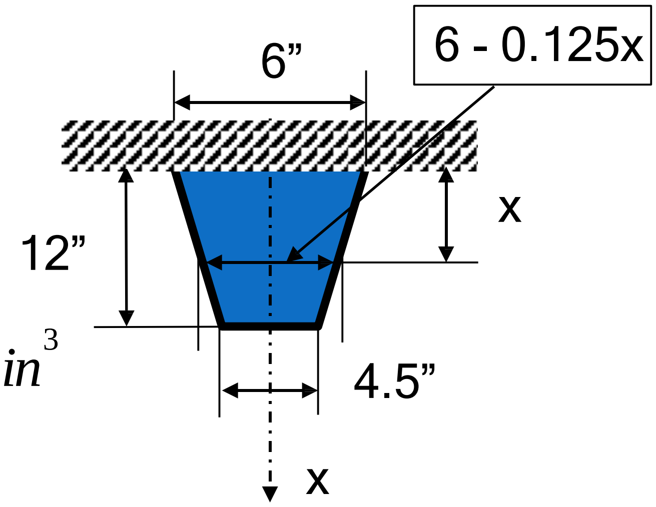
$$\underline{k}^{(1)} = \int_0^{12} \underline{B}^T E \underline{B} \; A dx = \frac{E}{(12)^2}\left(\int_0^{12} A(x)dx\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\int_0^{12} A(x)dx = \int_0^{12} t(6-0.125x)dx = t\int_0^{12}(6-0.125x)dx = 63\, in^3$$

$$\Rightarrow \underline{k}^{(1)} = \frac{E}{(12)^2}(63)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 13.125\times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Stiffness matrix of El #2

$$\underline{k}^{(2)} = \int_{12}^{24} \underline{B}^T E \underline{B} \; A dx = \frac{E}{(12)^2}\left(\int_{12}^{24} A(x)dx\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$



$$\int_{12}^{24} A(x)dx = \int_{12}^{24} t(6-0.125x)dx = t\int_{12}^{24}(6-0.125x)dx = 45\, in^3$$

$$\Rightarrow \underline{k}^{(2)} = \frac{E}{(12)^2}(45)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 9.375\times 10^6\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Now compute the element load vector due to distributed body force (weight)

$$\underline{f_b} = \int_{x_1}^{x_2} \underline{N}^T \; b \, dx$$

For element #1

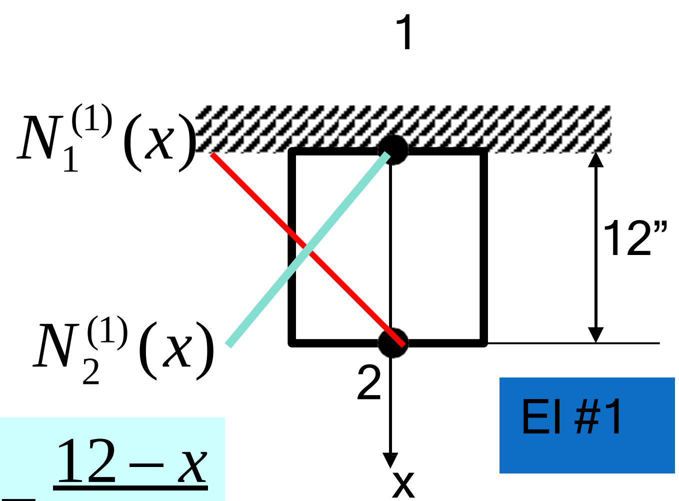
$$\underline{f_b}^{(1)} = \int_0^{12} \underline{N}^T \, b \, dx = \int_0^{12} \underline{N}^T (\rho A) \, dx$$

$$= \rho \int_0^{12} \underline{N}^T A \, dx$$

$$= \rho \int_0^{12} \begin{Bmatrix} N_1^{(1)}(x) \\ N_2^{(1)}(x) \end{Bmatrix} \underbrace{t(6 - 0.125x)}_{A(x)} \, dx$$

$$= 0.2836 \begin{Bmatrix} 33 \\ 30 \end{Bmatrix} lb$$

$$= \begin{Bmatrix} 9.3588 \\ 8.508 \end{Bmatrix} lb$$



$$N_1^{(1)}(x) = \frac{12 - x}{12}$$

$$N_2^{(1)}(x) = \frac{x}{12}$$

Superscript in parenthesis indicates element number

For element #2

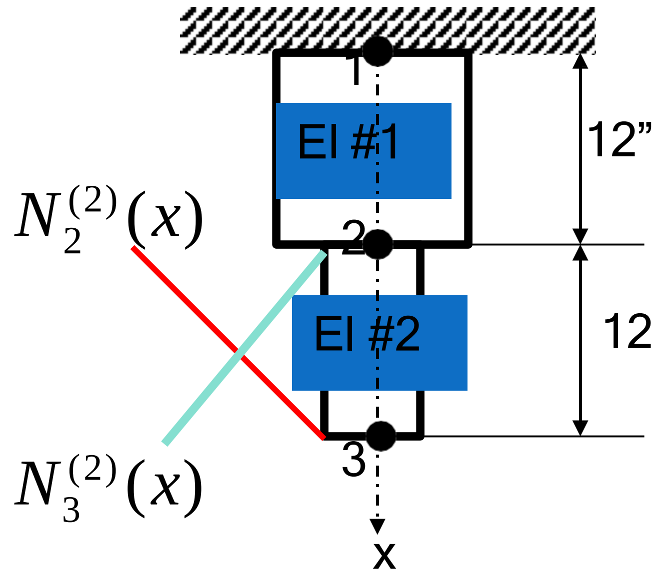
$$\underline{f}_b^{(2)} = \int_{12}^{24} \underline{\mathrm{N}}^T \ \mathrm{b\,dx} = \int_{12}^{24} \underline{\mathrm{N}}^T \ (\rho \mathrm{A})\,\mathrm{dx}$$

$$= \rho \int_{12}^{24} \underline{\mathrm{N}}^T \ A\,\mathrm{dx}$$

$$= \rho \int_{12}^{24} \begin{Bmatrix} N_2^{(2)}(x) \\ N_3^{(2)}(x) \end{Bmatrix} \underbrace{t(6-0.125x)}_{A(x)}\,\mathrm{dx}$$

$$= 0.2836 \begin{Bmatrix} 24 \\ 21 \end{Bmatrix} \ lb$$

$$= \begin{Bmatrix} 6.8064 \\ 5.9556 \end{Bmatrix} lb$$



$$N_2^{(2)}(x) = \frac{24-x}{12}$$

$$N_3^{(2)}(x) = \frac{x-12}{12}$$

Solution (2) Assemble the system equations

$$\underline{K} = 10^6 \times \begin{bmatrix} 13.125 & -13.125 & 0 \\ -13.125 & 22.5 & -9.375 \\ 0 & -9.375 & 9.375 \end{bmatrix}$$

$$\underline{f} = \underline{f_b} + \underline{f}_{concentrated\ load}$$

$$\underline{f}_b = \begin{Bmatrix} 9.3588 \\ 8.508 + 6.8064 \\ 5.9556 \end{Bmatrix} lb$$

$$\underline{f}_{concentrated\ load} = \begin{Bmatrix} 0 \\ 100 \\ 0 \end{Bmatrix} lb$$

$$\Rightarrow \underline{f} = \begin{Bmatrix} 9.3588 \\ 115.3144 \\ 5.9556 \end{Bmatrix} lb$$

Solution (3)

Hence we need to solve

$$10^6 \times \begin{bmatrix} 13.125 & -13.125 & 0 \\ -13.125 & 22.5 & -9.375 \\ 0 & -9.375 & 9.375 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} 9.3588 + R_1 \\ 115.3144 \\ 5.9556 \end{Bmatrix}$$

R_1 is the reaction at node 1.
Notice that since the boundary condition at x=0 (d_{1x}=0) has not been taken into account, the system matrix is not invertible.
Incorporating the boundary condition d_{1x}=0 we need to solve the following set of equations

$$10^6 \times \begin{bmatrix} 22.5 & -9.375 \\ -9.375 & 9.375 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} 115.3144 \\ 5.9556 \end{Bmatrix}$$

Solve to obtain

$$\begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} 0.92396\times10^{-5} \\ 0.98749\times10^{-5} \end{Bmatrix} in$$

Solution (4) Stress in elements

Notice that since we are using linear elements, the stress within each element is constant.

In element #1

$$\sigma^{(1)} = E\underline{B}^{(1)}\underline{d}^{(1)}$$

$$= \frac{E}{x_2 - x_1}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} \mathrm{d}_{1x} \\ \mathrm{d}_{2x} \end{Bmatrix}$$

$$= \frac{30\times10^6}{12}\mathrm{d}_{2x} \quad \because \mathrm{d}_{1x} = 0$$

$$= 23.099\ psi$$

In element #2

$$\sigma^{(2)} = E\underline{B}^{(2)}\underline{d}^{(2)}$$

$$= \frac{E}{x_3 - x_2}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix}$$

$$= \frac{30\times10^6}{12}\left(d_{3x} - d_{2x}\right)$$

$$= 1.5882\ psi$$

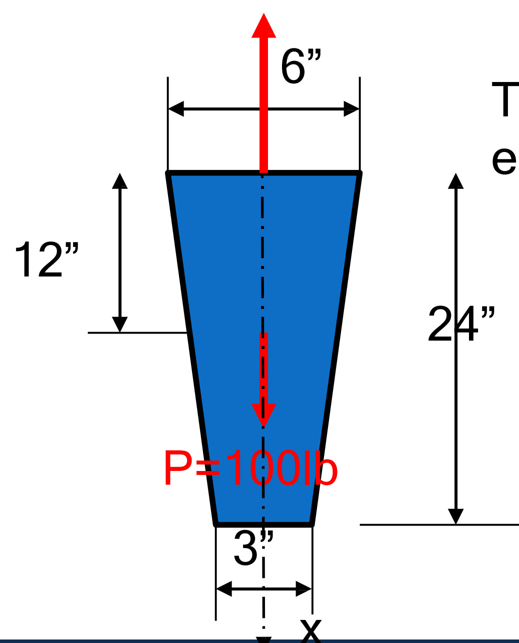
Solution (5) Reaction at support

Go back to the ***first line*** of the global equilibrium equations…

$$10^6 \times \begin{bmatrix} 13.125 & -13.125 & 0 \\ -13.125 & 22.5 & -9.375 \\ 0 & -9.375 & 9.375 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} 9.3588 + R_1 \\ 115.3144 \\ 5.9556 \end{Bmatrix}$$

$\Rightarrow R_1 = -130.6288\, lb$ (The -ve sign indicates that the force is in the -ve x-direction)

Check



The reaction at the wall from force equilibrium in the x-direction

$$R_1 = P + \int_{x=0}^{24} \rho A(x)\, dx$$

$$= 100 + \rho t \int_{x=0}^{24} (6 - 0.125x)\, dx$$

$$= 130.6288\, lb$$

Problem: Can you solve for the displacement and stresses analytically?

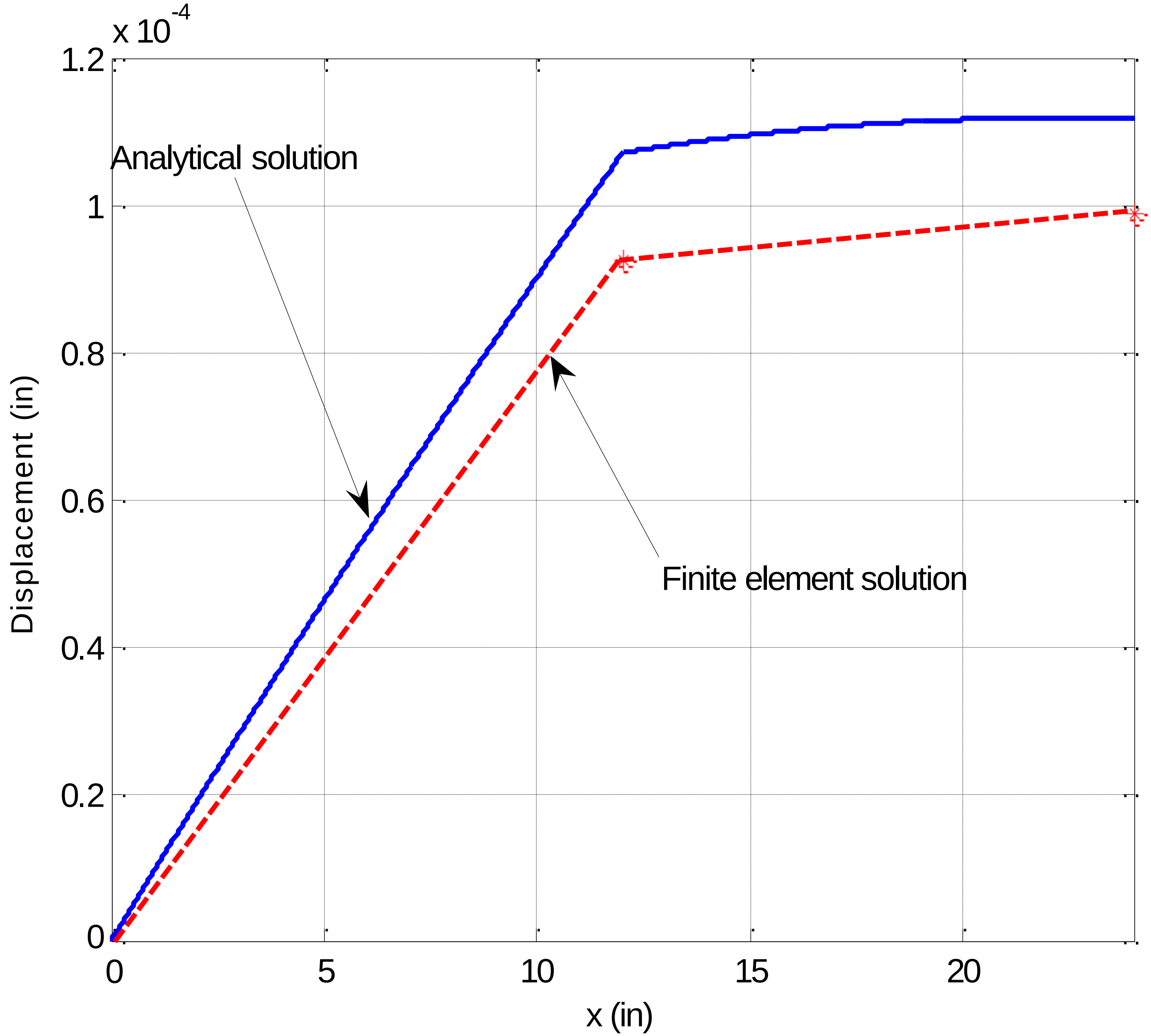
Check out

$$u_{anal} = \begin{cases} -4.727\times10^{-9}x^2 + 9.487\times10^{-7}x & for\ 0 \le x < 12 \\ -4.727\times10^{-9}x^2 + 2.0797\times10^{-7}x + 8.89\times10^{-6} & for\ 12 \le x \le 24 \end{cases}$$

Stress

$$\sigma(x)_{anal} = E\frac{du_{anal}}{dx} = 30\times10^6\frac{du_{anal}}{dx}$$

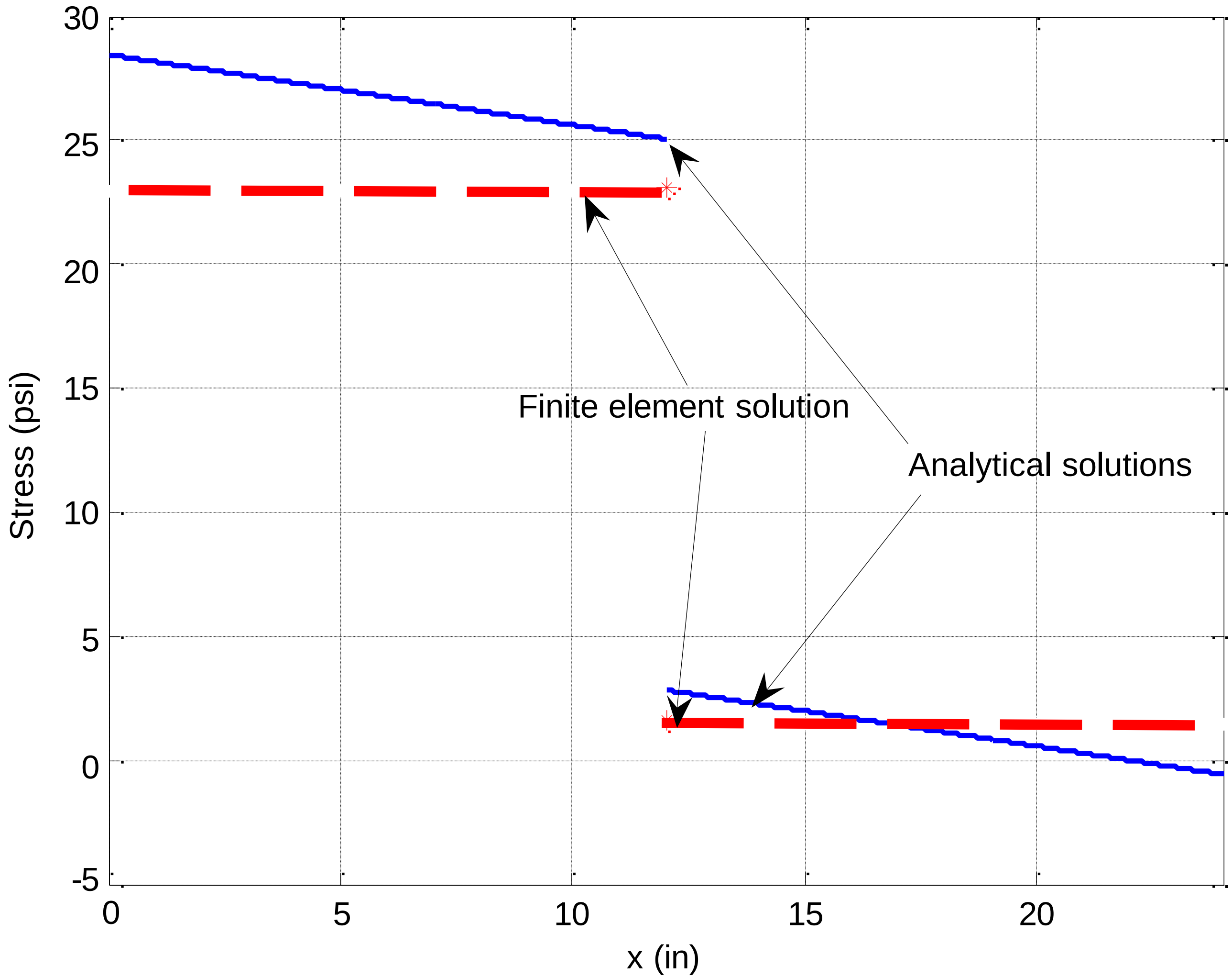
Comparison of displacement solutions



Notice:

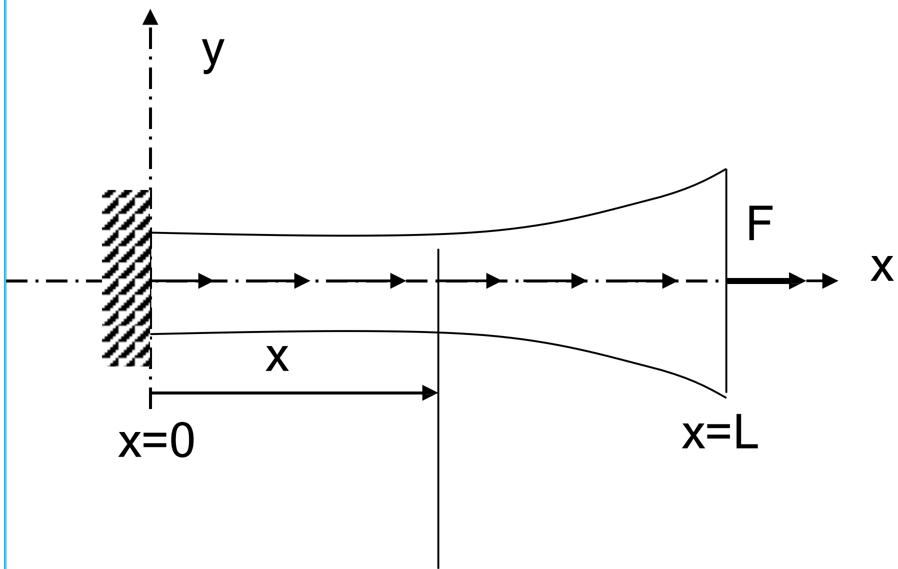
1. Slope discontinuity at x=12 (why?)
2. The finite element solution does not produce the exact solution even at the nodes
3. We may improve the solution by
   (1) Increasing the number of elements
   (2) Using higher order elements (e.g., quadratic instead of linear)

Comparison of stress solutions



The analytical as well as the finite element stresses are discontinuous across the elements

Axially loaded elastic bar



A(x) = cross section at x
b(x) = body force distribution (force per unit length)
E(x) = Young's modulus

Potential energy of the axially loaded bar corresponding to the exact solution u(x)

$$\Pi(u) = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - \int_0^L bu\ dx - Fu(x = L)$$

Finite element formulation, takes as its starting point, not the strong formulation, but the **Principle of Minimum Potential Energy**.

Task is to find the function 'w' that minimizes the potential energy of the system

$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\, dx - Fw(x = L)$$

From the Principle of Minimum Potential Energy, that function 'w' is the exact solution.

Rayleigh-Ritz Principle

Step 1. Assume a solution

$$w(x) = a_0\varphi_o(x) + a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots$$

Where $\varphi_o(x)$, $\varphi_1(x)$,... are "admissible" functions and a_o, a_1, etc are constants to be determined.

Step 2. Plug the approximate solution into the potential energy

$$\Pi(\mathrm{w}) = \frac{1}{2}\int_0^L \mathrm{EA}\left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2 \mathrm{dx} - \int_0^L \mathrm{bw\ dx} - \mathrm{Fw}(\mathrm{x} = \mathrm{L})$$

Step 3. Obtain the coefficients a_o, a_1, etc by setting

$$\frac{\partial \Pi(\mathrm{w})}{\partial a_i} = 0, \quad i = 0,1,2,\ldots$$

The approximate solution is

$$u(x) = a_0\varphi_o(x) + a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots$$

Where the coefficients have been obtained from step 3

Need to find a systematic way of choosing the approximation functions.

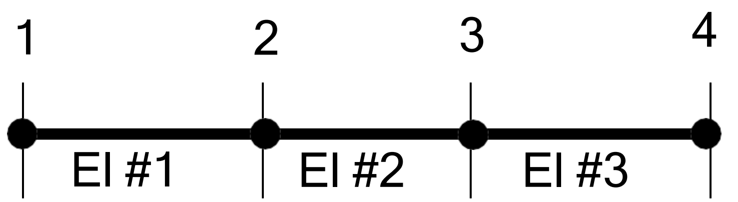
One idea: Choose polynomials!

$w(x) = a_0$ Is this good? (Is ‘1’ an “admissible” function?)

$w(x) = a_1 x$ Is this good? (Is ‘x’ an “admissible” function?)

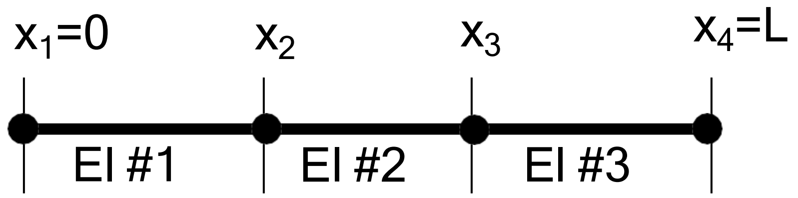
Finite element idea:

Step 1: Divide the truss into **finite elements** connected to each other through special points ("**nodes**")



Total potential energy=sum of potential energies of the elements

$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\ dx - Fw(x = L)$$



Total potential energy

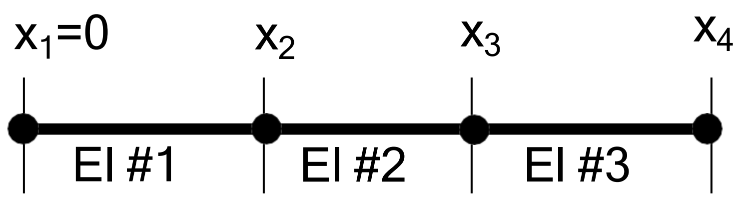
$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\,dx - Fw(x = L)$$

Potential energy of element 1:

$$\Pi_1(w) = \frac{1}{2}\int_{x_1}^{x_2} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_1}^{x_2} bw\,dx$$

Potential energy of element 2:

$$\Pi_2(w) = \frac{1}{2}\int_{x_2}^{x_3} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_2}^{x_3} bw\,dx$$



Potential energy of element 3:

$$\Pi_3(w) = \frac{1}{2}\int_{x_3}^{x_4} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_3}^{x_4} bw\, dx - Fw(x = L)$$

Total potential energy=sum of potential energies of the elements

$$\Pi(w) = \Pi_1(w) + \Pi_2(w) + \Pi_3(w)$$

Step 2: Describe the behavior of each element

Recall that in the "**direct stiffness**" approach for a bar element, we derived the stiffness matrix of each element directly (See lecture on Trusses) using the following steps:

TASK 1: Approximate the displacement within each bar as a straight line
TASK 2: Approximate the strains and stresses and realize that a bar (with the approximation stated in Task 1) is exactly like a spring with k=EA/L
TASK 3: Use the principle of **force equilibrium** to generate the stiffness matrix

Now, we will show you a systematic way of deriving the stiffness matrix (sections 2.2 and 3.1 of Logan).

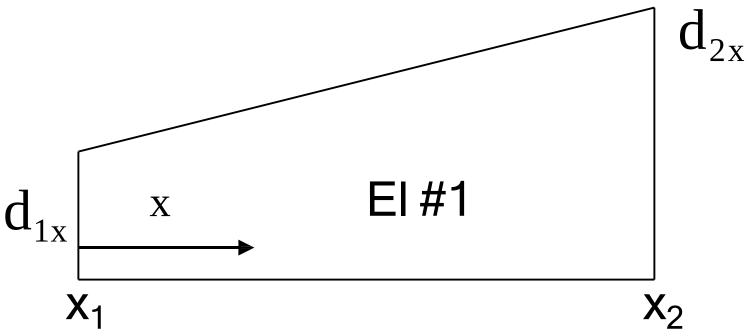
TASK 1: APPROXIMATE THE DISPLACEMENT WITHIN EACH ELEMENT
TASK 2: APPROXIMATE THE STRAIN and STRESS WITHIN EACH ELEMENT
TASK 3: DERIVE THE STIFFNESS MATRIX OF EACH ELEMENT (next class) USING THE PRINCIPLE OF MIN. POT ENERGY

Notice that the first two tasks are similar in the two methods. The only difference is that now we are going to use the principle of minimum potential energy, rather than force equilibrium, to derive the stiffness matrix.

TASK 1: APPROXIMATE THE DISPLACEMENT WITHIN EACH ELEMENT

Simplest assumption: displacement varying linearly inside each bar

$$w(x) = a_0 + a_1 x$$



How to obtain a_0 and a_1?

$$w(x_1) = a_0 + a_1 x_1 = d_{1x}$$
$$w(x_2) = a_0 + a_1 x_2 = d_{2x}$$

$$w(x_1) = a_0 + a_1 x_1 = d_{1x}$$

$$w(x_2) = a_0 + a_1 x_2 = d_{2x}$$

Solve simultaneously

$$a_0 = \frac{x_2}{x_2 - x_1} d_{1x} - \frac{x_1}{x_2 - x_1} d_{2x}$$

$$a_1 = -\frac{1}{x_2 - x_1} d_{1x} + \frac{1}{x_2 - x_1} d_{2x}$$

Hence

$$w(x) = a_0 + a_1 x = \frac{x_2 - x}{x_2 - x_1} d_{1x} + \frac{x - x_1}{x_2 - x_1} d_{2x} = N_1(x) d_{1x} + N_2(x) d_{2x}$$

$N_1(x)$ $N_2(x)$

“**Shape functions**” $N_1(x)$ and $N_2(x)$

In matrix notation, we write

$$\boxed{w(x) = \underline{N}\,\underline{d}} \qquad (1)$$

Vector of nodal shape functions

$$\underline{N} = \begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix} = \begin{bmatrix} \dfrac{x_2 - x}{x_2 - x_1} & \dfrac{x - x_1}{x_2 - x_1} \end{bmatrix}$$

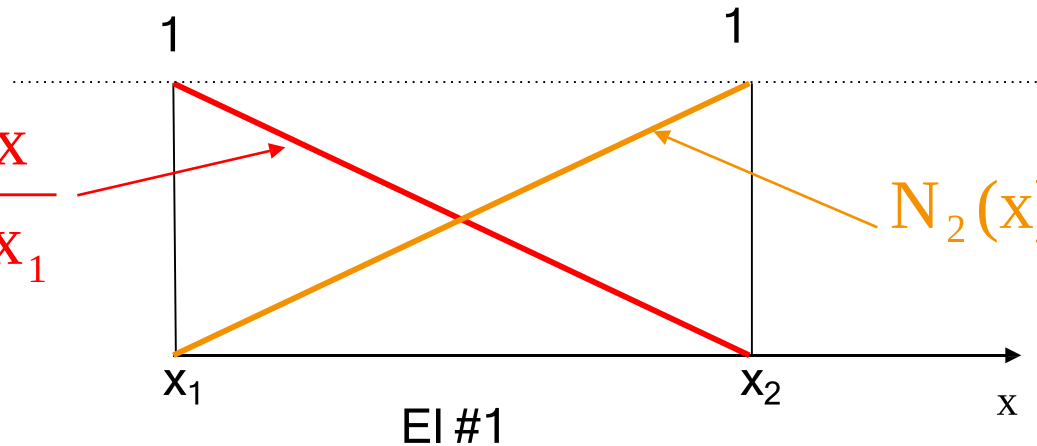
Vector of nodal displacements

$$\underline{d} = \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

NOTES: PROPERTIES OF THE SHAPE FUNCTIONS

1. <u>Kronecker delta property</u>: The shape function at any node has a value of 1 at that node and a value of zero at ALL other nodes.

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$



$$N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

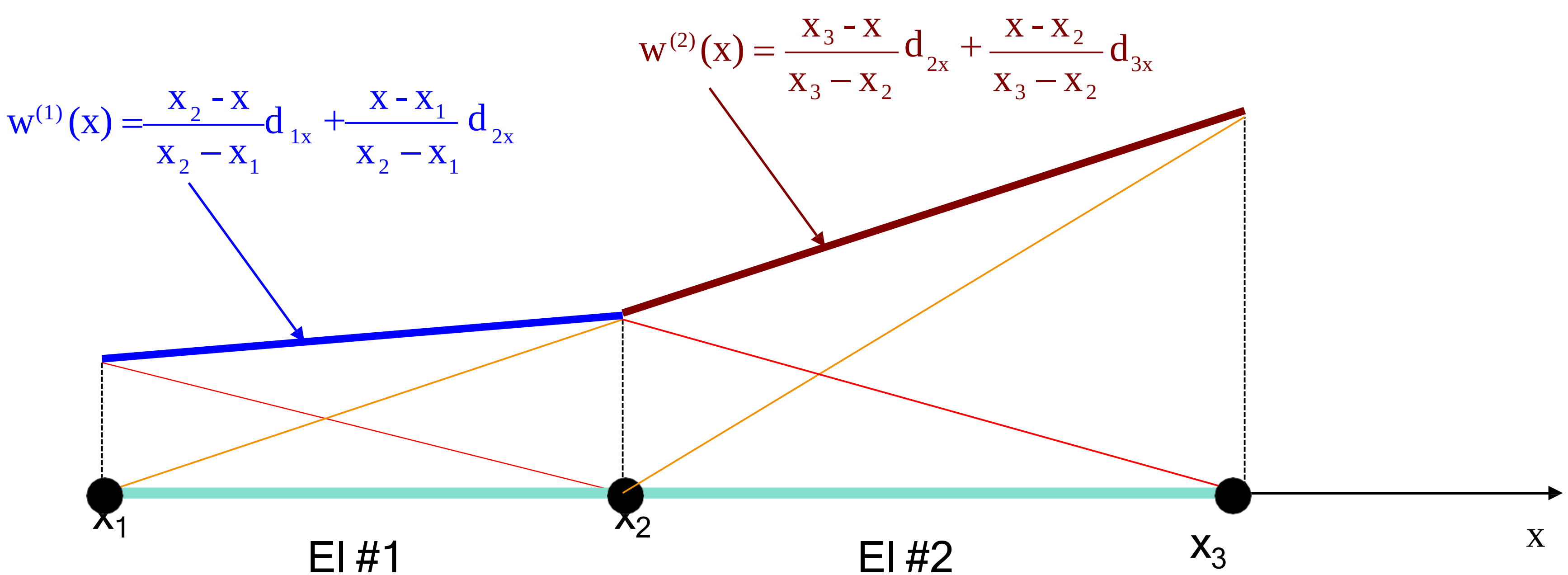
Check

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$

$$\Rightarrow N_1(x = x_1) = \frac{x_2 - x_1}{x_2 - x_1} = 1$$

$$\text{and } N_1(x = x_2) = \frac{x_2 - x_2}{x_2 - x_1} = 0$$

2. **Compatibility:** The displacement approximation is **continuous across element boundaries**



At $x=x_2$

$$w^{(1)}(x=x_2)=\frac{x_2-x_2}{x_2-x_1}d_{1x}+\frac{x_2-x_1}{x_2-x_1}d_{2x}=d_{2x}$$

$$w^{(2)}(x=x_2)=\frac{x_3-x_2}{x_3-x_2}d_{2x}+\frac{x_2-x_2}{x_3-x_2}d_{3x}=d_{2x}$$

Hence the displacement approximation is continuous across elements

3. Completeness

$$N_1(x) + N_2(x) = 1 \quad \text{for all } x$$
$$N_1(x)x_1 + N_2(x)x_2 = x \quad \text{for all } x$$

Use the expressions

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1};$$

$$N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

And check

$$N_1(x) + N_2(x) = \frac{x_2 - x}{x_2 - x_1} + \frac{x - x_1}{x_2 - x_1} = 1$$

$$\text{and } N_1(x)\ x_1 + N_2(x)\ x_2 = \frac{x_2 - x}{x_2 - x_1} x_1 + \frac{x - x_1}{x_2 - x_1} x_2 = x$$

Rigid body mode

$$N_1(x) + N_2(x) = 1 \qquad \text{for all } x$$

What do we mean by "**rigid body modes**"?

Assume that $d_{1x}=d_{2x}=1$, this means that the element should translate in the positive x direction by 1. Hence ***ANY point*** (x) on the bar should have unit displacement. Let us see whether the displacement approximation allows this.

$$w(x) = N_1(x)d_{1x} + N_2(x)d_{2x} = N_1(x) + N_2(x) = 1$$

YES!

Constant strain states

$$N_1(x)x_1 + N_2(x)x_2 = x \quad \text{at all } x$$

<u>What do we mean by "**constant strain states**"?</u>

Assume that $d_{1x}=x_1$ and $d_{2x}=x_2$. The strain at ***<u>ANY point</u>*** (x) within the bar is

$$\varepsilon(x) = \frac{d_{2x} - d_{1x}}{x_2 - x_1} = \frac{x_2 - x_1}{x_2 - x_1} = 1$$

Let us see whether the displacement approximation allows this.

$$w(x) = N_1(x)d_{1x} + N_2(x)d_{2x} = N_1(x)x_1 + N_2(x)x_2 = x$$

$$\text{Hence, } \varepsilon(x) = \frac{dw(x)}{dx} = 1$$

YES!

Completeness = Rigid body modes + Constant Strain states

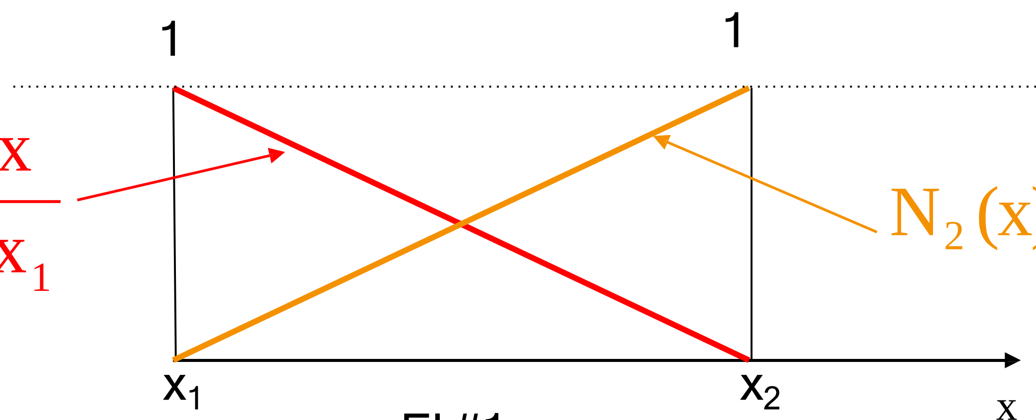
Compatibility + Completeness $\Rightarrow$ Convergence

Ensure that the solution gets better as more elements are introduced and, in the limit, approaches the exact answer.

4. How to write the expressions for the shape functions easily (without having to derive them each time):

Start with the **Kronecker delta property** (the shape function at any node has value of 1 at that node and a value of zero at all other nodes)

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$



El #1

$$N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

Node at which N_1 is 0

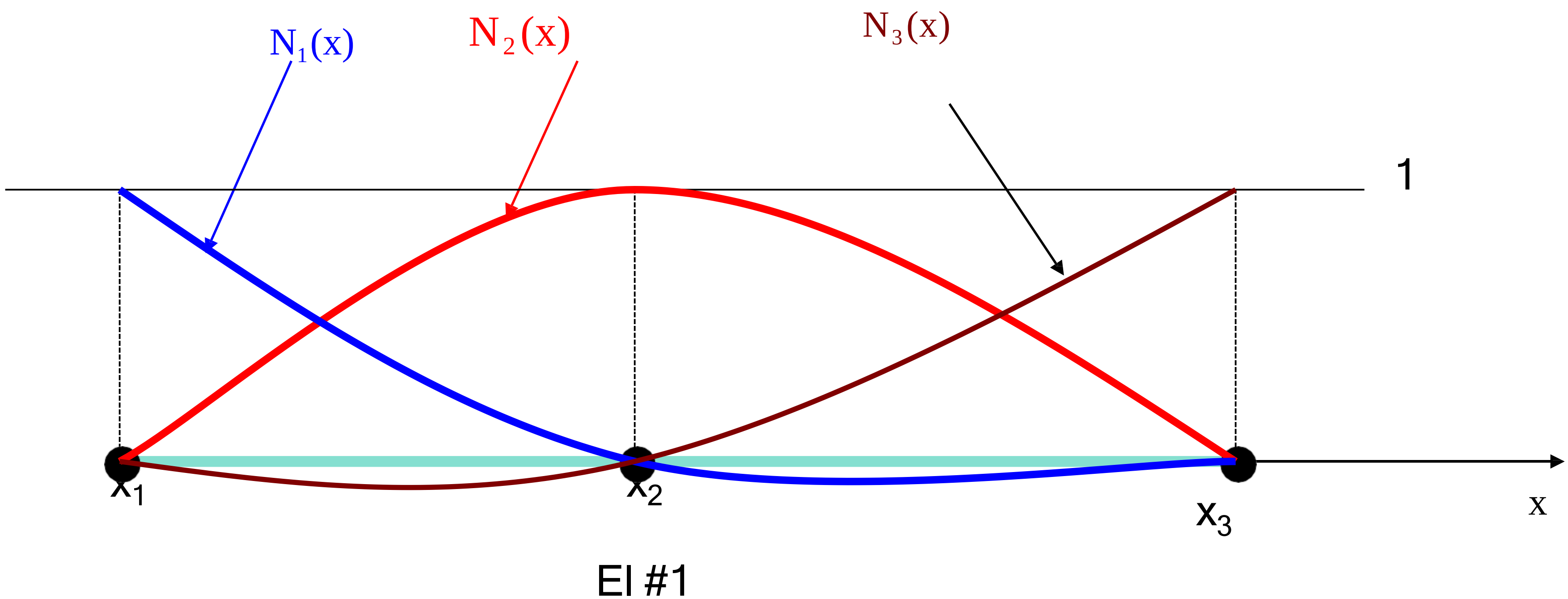
$$N_1(x) = \frac{(x_2 - x)}{(x_2 - x_1)}$$

Notice that the length of the element = $x_2 - x_1$

$$N_2(x) = \frac{(x_1 - x)}{(x_1 - x_2)} = \frac{(x - x_1)}{(x_2 - x_1)}$$

The denominator is the numerator evaluated at the node itself

A slightly fancier assumption:
displacement varying **quadratically** inside each bar



$$N_1(x) = \frac{(x_2 - x)(x_3 - x)}{(x_2 - x_1)(x_3 - x_1)}$$

$$N_2(x) = \frac{(x_1 - x)(x_3 - x)}{(x_1 - x_2)(x_3 - x_2)}$$

$$N_3(x) = \frac{(x_1 - x)(x_2 - x)}{(x_1 - x_3)(x_2 - x_3)}$$

$$w(x) = N_1(x)d_{1x} + N_2(x)d_{2x} + N_3(x)d_{3x}$$

This is a **quadratic finite element** in 1D and it has three nodes and three associated shape functions per element.

TASK 2: APPROXIMATE THE STRAIN and STRESS WITHIN EACH ELEMENT

From equation (1), the displacement within each element

$$w(x) = \underline{N}\,\underline{d}$$

Recall that the **strain** in the bar $\varepsilon = \frac{dw}{dx}$

Hence

$$\varepsilon = \left[\frac{d\underline{N}}{dx}\right]\underline{d} = \underline{B}\,\underline{d} \qquad (2)$$

The matrix $\underline{B}$ is known as the "**strain-displacement matrix**"

$$\underline{B} = \left[\frac{d\underline{N}}{dx}\right]$$

For a linear finite element

$$\underline{N} = \left[N_1(x) \quad N_2(x) \right] = \left[\frac{x_2 - x}{x_2 - x_1} \quad \frac{x - x_1}{x_2 - x_1} \right]$$

Hence

$$\underline{B} = \left[\frac{-1}{x_2 - x_1} \quad \frac{1}{x_2 - x_1} \right] = \frac{1}{x_2 - x_1} \left[-1 \quad 1 \right]$$

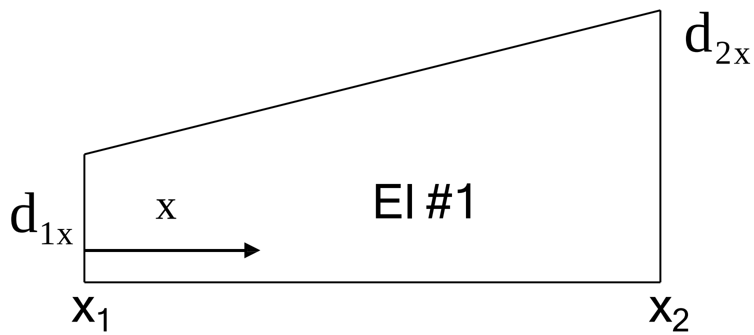
$$\varepsilon = \underline{B}\ \underline{d} = \left[\frac{-1}{x_2 - x_1} \quad \frac{1}{x_2 - x_1} \right] \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

$$= \frac{d_{2x} - d_{1x}}{x_2 - x_1}$$

Hence, strain is a **constant** within each element (only for a linear element)!

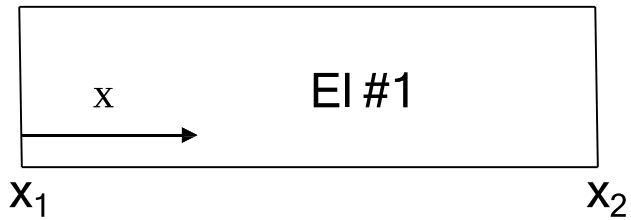
Displacement is linear

$$w(x) = a_0 + a_1 x$$



Strain is constant

$$\varepsilon = \frac{d_{2x} - d_{1x}}{x_2 - x_1}$$



Recall that the **stress** in the bar $$\sigma = E\varepsilon = E\frac{du}{dx}$$

Hence, inside the element, the approximate stress is

$$\boxed{\sigma = E\underline{B}\,\underline{d}} \qquad (3)$$

For a linear element the stress is also constant inside each element. This has the implication that the stress (and strain) is **discontinuous across element boundaries** in general.

Summary

Inside an element, the three most important approximations **in terms of the nodal displacements** ($\underline{d}$) are:

Displacement approximation in terms of shape functions

$$u(x) = \underline{N}\,\underline{d} \qquad (1)$$

Strain approximation in terms of strain-displacement matrix

$$\varepsilon(x) = \underline{B}\,\underline{d} \qquad (2)$$

Stress approximation in terms of strain-displacement matrix and Young's modulus

$$\sigma = E\underline{B}\,\underline{d} \qquad (3)$$

Summary

For a **linear element**

Displacement approximation in terms of shape functions

$$u(x) = \begin{bmatrix} \dfrac{x_2 - x}{x_2 - x_1} & \dfrac{x - x_1}{x_2 - x_1} \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

Strain approximation

$$\varepsilon = \frac{1}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

Stress approximation

$$\sigma = \frac{E}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$